# Reduced-order models for flow control

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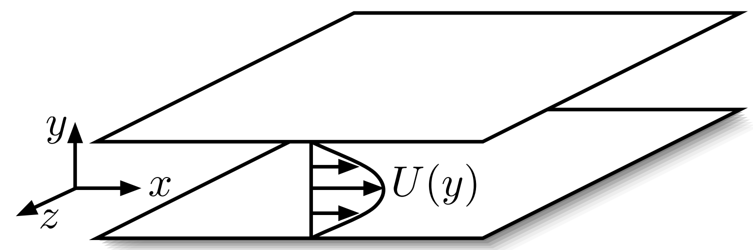
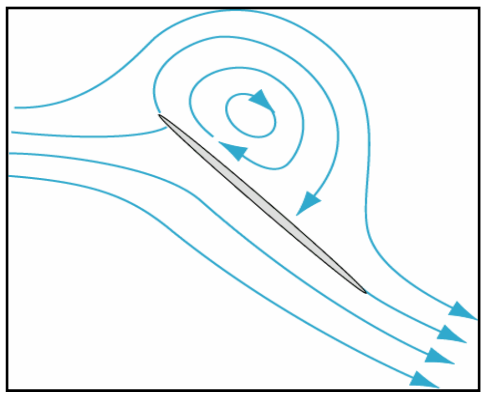
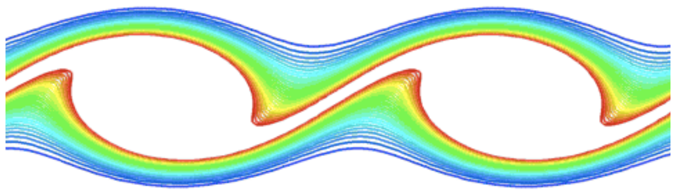
# Outline

- Reduced-order models: POD and balanced truncation
  - Importance of inner product for Galerkin projection
  - Balanced truncation
  - Method of snapshots
- Applications
  - Linearized channel flow
  - Separating flow past an airfoil
- Dynamically scaling POD modes
  - Free shear layer
  - Scaled basis functions
  - Template fitting
  - Equations for the shear layer thickness



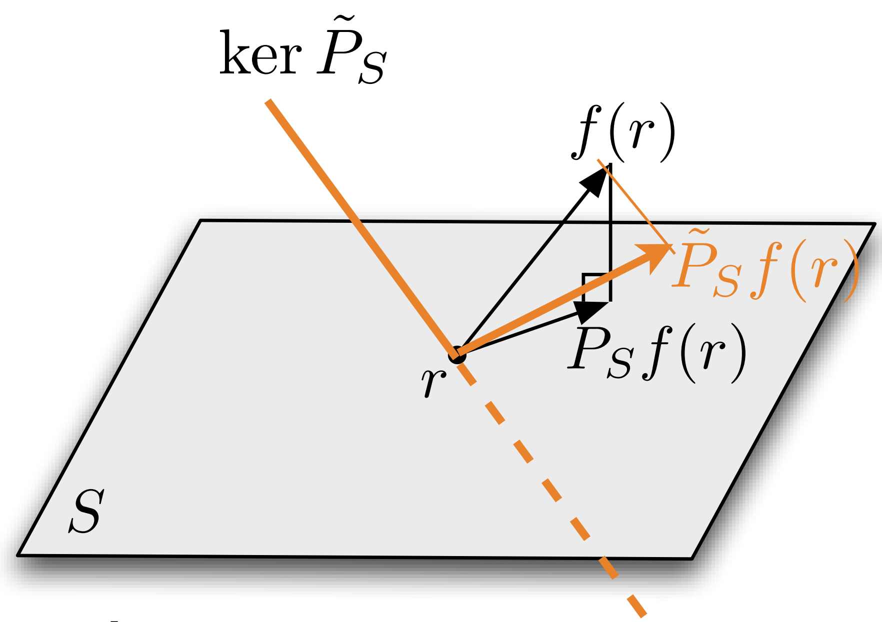


# Galerkin projection

- Dynamics evolve on a high-dimensional space (or infinite-dim'l)
- Project dynamics onto a low-dimensional subspace S

$$\dot{x} = f(x) \qquad x \in V$$

$$r \in S \subset V$$




- Define dynamics on the subspace by

$$\dot{r} = P_S f(r) \qquad P_S : V \to S$$ is a projection

- Two choices:
  - choice of subspace
  - choice of inner product
    (equivalently, choice of the nullspace for a non-orthogonal projection)

# Proper Orthogonal Decomposition (POD)

- Obtain "optimal" basis for the subspace, from data
  - Gather data, as "snapshots" u(x,t) from simulations or experiments
  - Determine orthonormal basis functions that optimally span the data:

$$P_n u(x,t) = \sum_{j=1}^{n} a_j(t)\varphi_j(x)$$

  - Minimize $\displaystyle\int_0^T \|u(t) - P_n u(t)\|^2\,dt$

$\varphi_j \in V$

POD modes

$S = \operatorname{span}\{\varphi_j\}$

  - Solution: SVD of the matrix of snapshots
- Limitations
  - Optimal for capturing a given dataset, not necessarily dynamics
  - Low-energy modes may be important to the dynamics
  - POD says nothing about which inner product you should use

# Energy-based inner products

- Reduced-order models can behave unpredictably
  - Can even change stability type of equilibria!
    [Rempfer, Thoret. CFD 2000]
- Energy-based inner products behave better
  - Consider a system with a stable equilibrium point at the origin
  - An energy-based inner product induces a norm that is a Liapunov function:

$$\langle x, y \rangle = x^T Q y, \qquad Q > 0 \qquad \begin{aligned} &V(x) = x^T Q x \text{ is a Liapunov function} \\ &\dot{V}(x) = 2x^T f(x) \le 0, \qquad \forall x \in U \end{aligned}$$

  - Useful result: if such an inner product is used for Galerkin projection, the reduced-order model is guaranteed to have the same stability type as the full system [Rowley, Colonius, Murray, Phys D 2004]
  - One interpretation of balanced truncation: use adjoint simulations to determine an appropriate inner product (the "observability Gramian," always a Lyapunov function)

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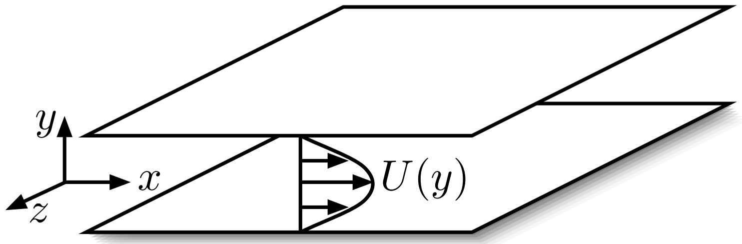
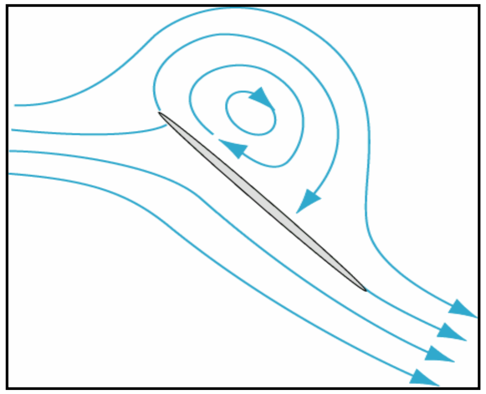
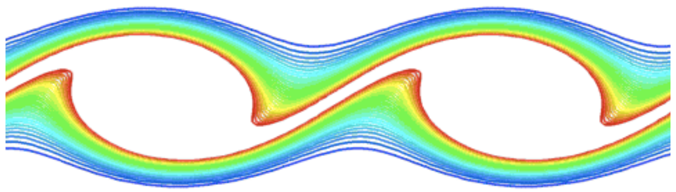
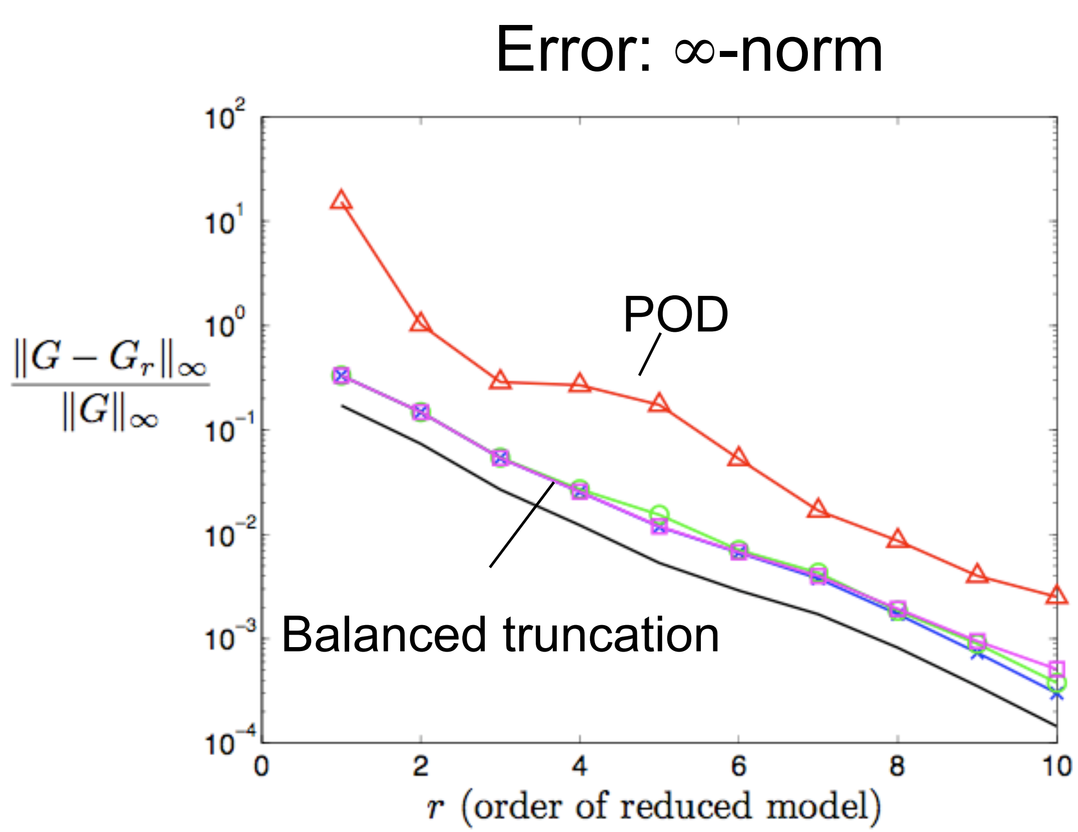




# Are POD modes optimal?

- POD modes are not optimal for Galerkin projection
  - POD determines a subspace that optimally captures the energy in a given dataset
  - These modes are usually not optimal for Galerkin projection
    - Low-energy modes can play an important role in the dynamics [Aubry, Holmes, Lumley, 1988; Smith 2002 PhD thesis, Princeton]
  - Can often do better with balanced truncation [Moore 1981]

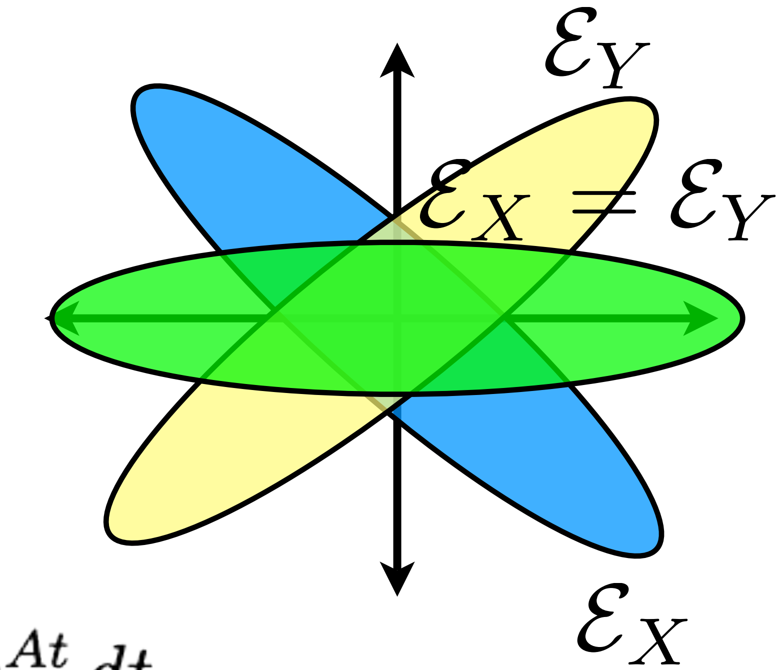


# Balanced truncation

- Why doesn't everybody use this?
  - Valid for stable, linear systems
    - Extensions for unstable systems [Jonckheere & Silverman 1983, Zhou 2001]
    - Extensions for nonlinear systems [Scherpen 1993, Lall, Marsden, Glavaski 1999]
  - Computationally expensive for large systems
    - $n^3$ computational time: $n > 10^5$ for typical fluids simulations
- Improvements for large systems
  - POD is tractable for large systems. Can we extend, e.g., the method of snapshots, to compute balancing transformations?
  - Based on earlier snapshot-based methods:

    Lall, Marsden, & Glavaski, 1999
    Willcox & Peraire, 2001

# Overview of balanced truncation




- Start with a stable, linear input-output system

What are you interested in capturing?

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

- Compute controllability and observability Gramians

$$X = \int_0^\infty e^{At} BB^* e^{A^* t}\, dt \qquad Y = \int_0^\infty e^{A^* t} C^* C e^{At}\, dt$$
$$AX + XA^* + BB^* = 0 \qquad A^* Y + YA + C^* C = 0$$

States easily excited by an input

States that have large influence on the output

- Find a transformation $T$ that simultaneously diagonalizes $X$ and $Y$

$$x = Tz, \qquad T^{-1} X (T^{-1})^* = T^* Y T = \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

- Change coordinates, and truncate states that are least controllable/observable

# Empirical Gramians

- Construct Gramians from impulse response data
  - Not solving Liapunov equations
  - For a single input: compute impulse-state response:

    $$\dot{x} = Ax, \qquad x(0) = B$$

    solution

    $$x(t) = e^{At}B$$

  - The controllability Gramian is then $$W_c = \int_0^\infty x(t)x(t)^T\,dt$$
  - Discretize in time, collect snapshots into a matrix:

    $$X = \begin{bmatrix} | & & | \\ x(t_1) & \cdots & x(t_m) \\ | & & | \end{bmatrix}$$

  - Then $\boxed{W_c \approx XX^T}$
  - For observability Gramian, same procedure, but use adjoint equations $\dot{z} = A^* z \qquad z(0) = C^*$
  - For multiple inputs/outputs, same procedure, but do one impulse-response for each input/output

[Lall et al, 1999]

# Large numbers of outputs

- Often, we are interested in modeling the full state
  - If dimension is large, project output onto POD modes
  - POD gives optimally-close output-projected system (in 2-norm)

u

$$\dot{x} = Ax + Bu$$
$$y = x$$

Full state

Original system, full dynamics

Output projection

u

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

POD coefficients

Output-projected system, full dynamics

Balanced truncation

u

$$\dot{a} = \Psi^T A \Phi a + \Psi^T B u$$
$$y = C\Phi a$$

POD coefficients

Reduced-order model

# Approximate balanced truncation for large systems

- Method of snapshots enables one to compute approximate balanced truncations with cost similar to POD
  - One simulation for each control input, one adjoint simulation for each output
  - One SVD, (# direct snapshots) x (# adjoint snapshots)
  - If number of outputs is large, method for projection onto smaller-rank output
- Balanced truncation is just POD with respect to an inner product defined by the observability Gramian Y:

$$\langle x_1, x_2 \rangle_Y = x_1^T Y x_2$$

- Observability Gramian is always a Liapunov function => preserves stability!
- Obtain set of bi-orthogonal modes:

direct modes: $\{\varphi_1, \dots, \varphi_n\}$

adjoint modes: $\{\psi_1, \dots, \psi_n\}$

bi-orthogonal: $\langle \psi_i, \varphi_j \rangle = \delta_{ij}$

Galerkin:

$$\dot{x} = f(x)$$

$$x(t) = \sum_j a_j(t)\varphi_j$$

$$\dot{a}_j(t) = \langle \psi_j, f(x) \rangle$$

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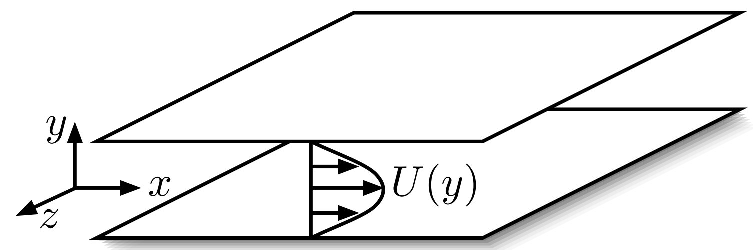
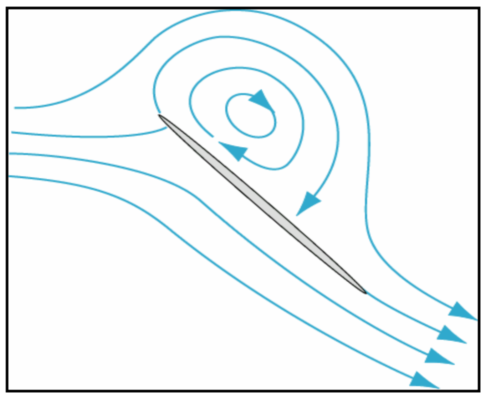
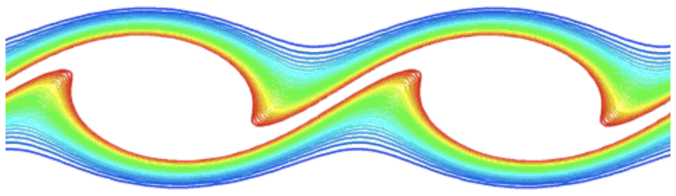
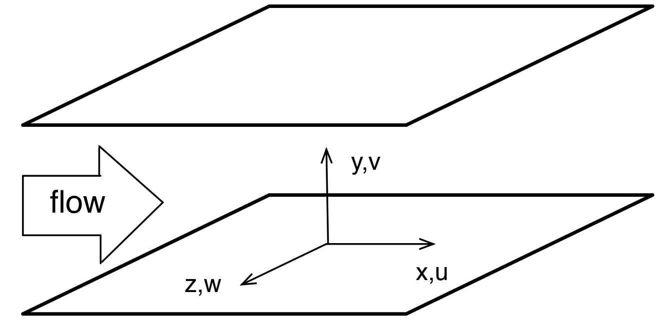


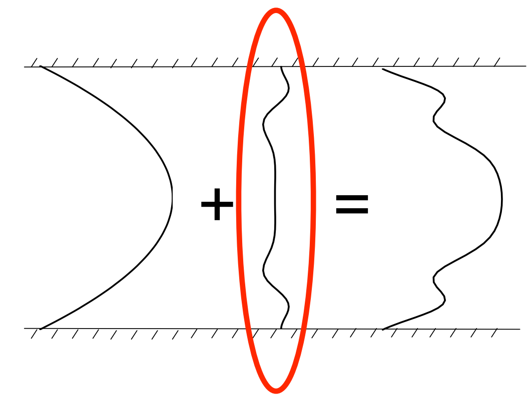


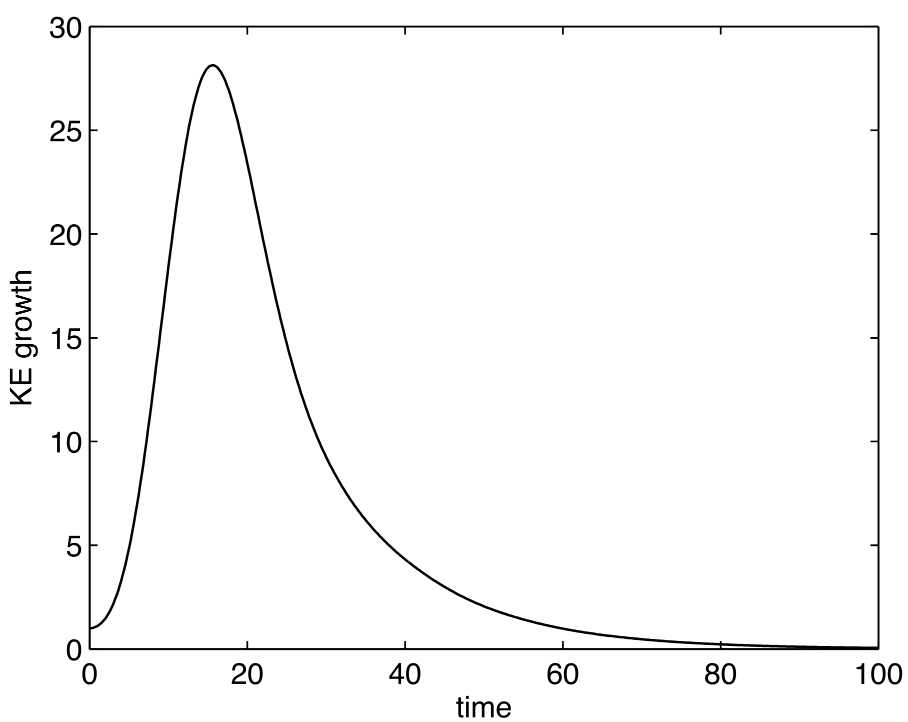
# Application: Linearized Channel Flow

- Plane channel flow with periodic boundary conditions
  - Goal: delay transition to turbulence using feedback control
  - Goal: improved understanding of transition mechanisms
  - Focus: low-dimensional models of transition




- Linear development of small perturbations
  - Transition not predicted correctly by linear stability theory
  - Non-normality of the governing operator results in large transient growth, even though linearized flow is stable
  - Large linear system with complex dynamic behavior

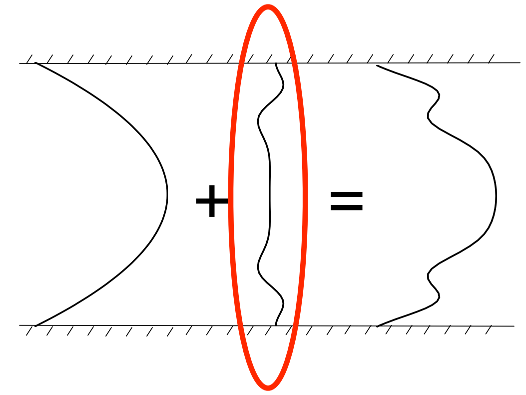



Previous work:
Trefethen et al [Science, 1993]
Farrell & Ioannou [96,96,01]
Schmid & Henningson [01]
Bamieh & Jovanovic [01,03]

# Governing Equations

- Navier-Stokes equations linearized about a laminar profile
- Perturbation dynamics fully described by wall-normal velocity v and wall-normal vorticity η
- Clamped boundary conditions $v(\pm 1) = \dfrac{\partial v}{\partial y}(\pm 1) = 0$




Orr-Sommerfeld/Squire system

$$\frac{\partial}{\partial t}\begin{bmatrix} -\Delta & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} L_{OS} & 0 \\ -U'\partial_z & L_{SQ} \end{bmatrix}\begin{bmatrix} v \\ \eta \end{bmatrix}$$

$$L_{OS} = U\partial_x\Delta - U''\partial_x - \frac{1}{Re}\Delta^2$$

$$L_{SQ} = -U\partial_x + \frac{1}{Re}\Delta$$

Adjoint system

$$\frac{\partial}{\partial t}\begin{bmatrix} -\Delta & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} L^*_{OS} & U'\partial_z \\ 0 & L^*_{SQ} \end{bmatrix}\begin{bmatrix} v \\ \eta \end{bmatrix}$$

$$L^*_{OS} = -U\partial_x\Delta - 2U'\partial_x\partial_y - \frac{1}{Re}\Delta^2$$

$$L^*_{SQ} = U\partial_x + \frac{1}{Re}\Delta$$

actuation disturbances

$$\dot{x} = Ax + Bu_1 + Fu_2$$

- System in standard state-space form with actuation and disturbances

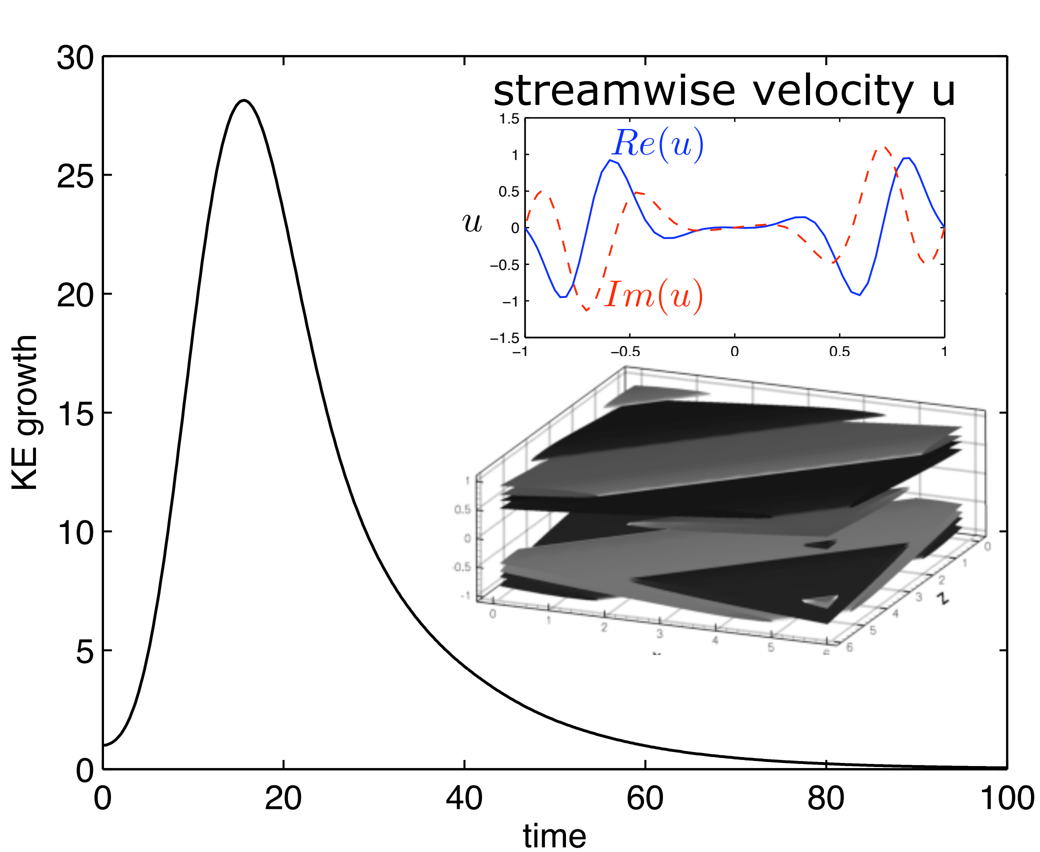
Analysis as an input-output system: Bamieh & Jovanovic [01,03]

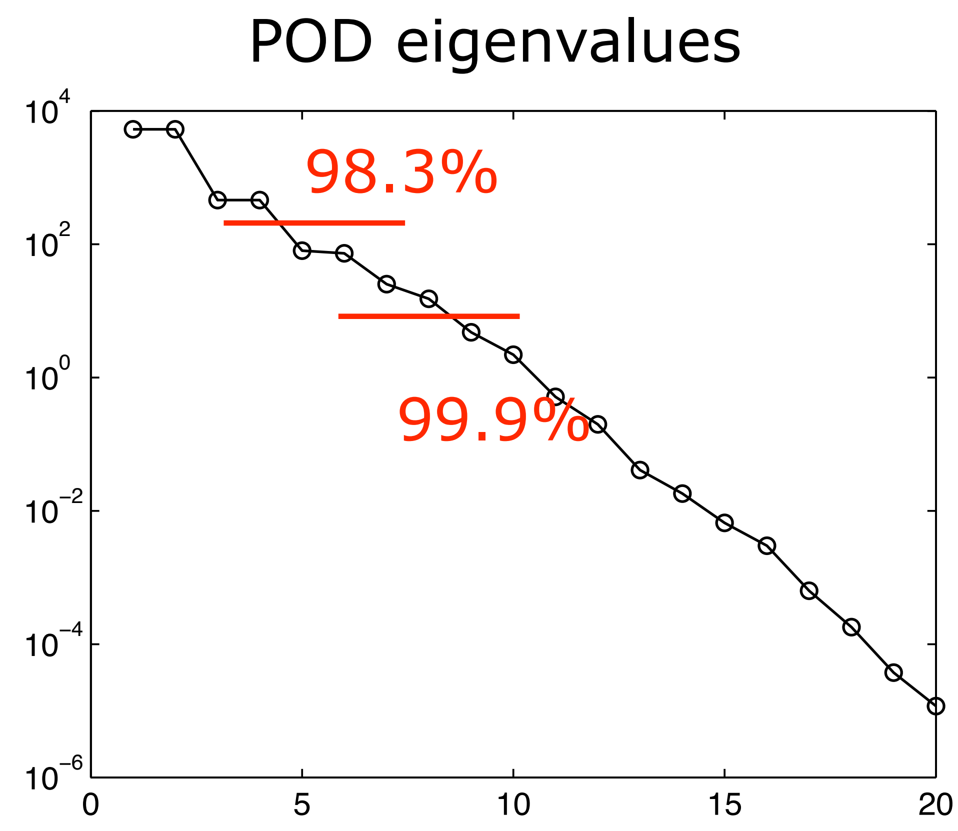
# Single-wavenumber perturbation - optimal

- Perturbations of the form

$$q = \hat{q}(y)e^{i\alpha x + i\beta z + \lambda t} \qquad q = \begin{bmatrix} v \\ \eta \end{bmatrix}$$

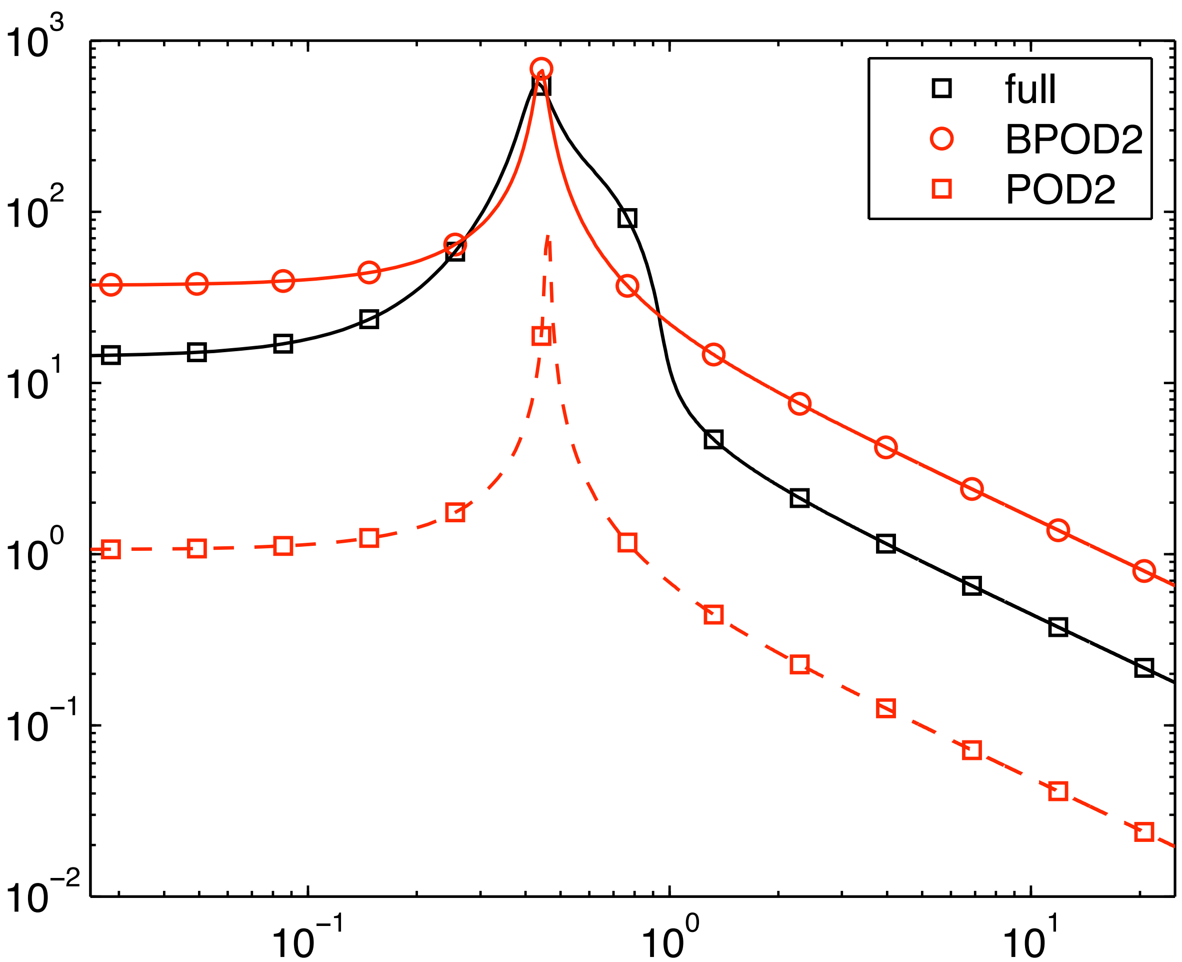
- System can be analyzed in 1-D so that full balanced truncation is tractable, allowing comparison with the BPOD approximation and POD
- Well-studied cases (Farrell, Henningson, Reddy, Schmid, Jovanovic, Bamieh)
- Case presented here α=1, β=1 and exhibits rich dynamics

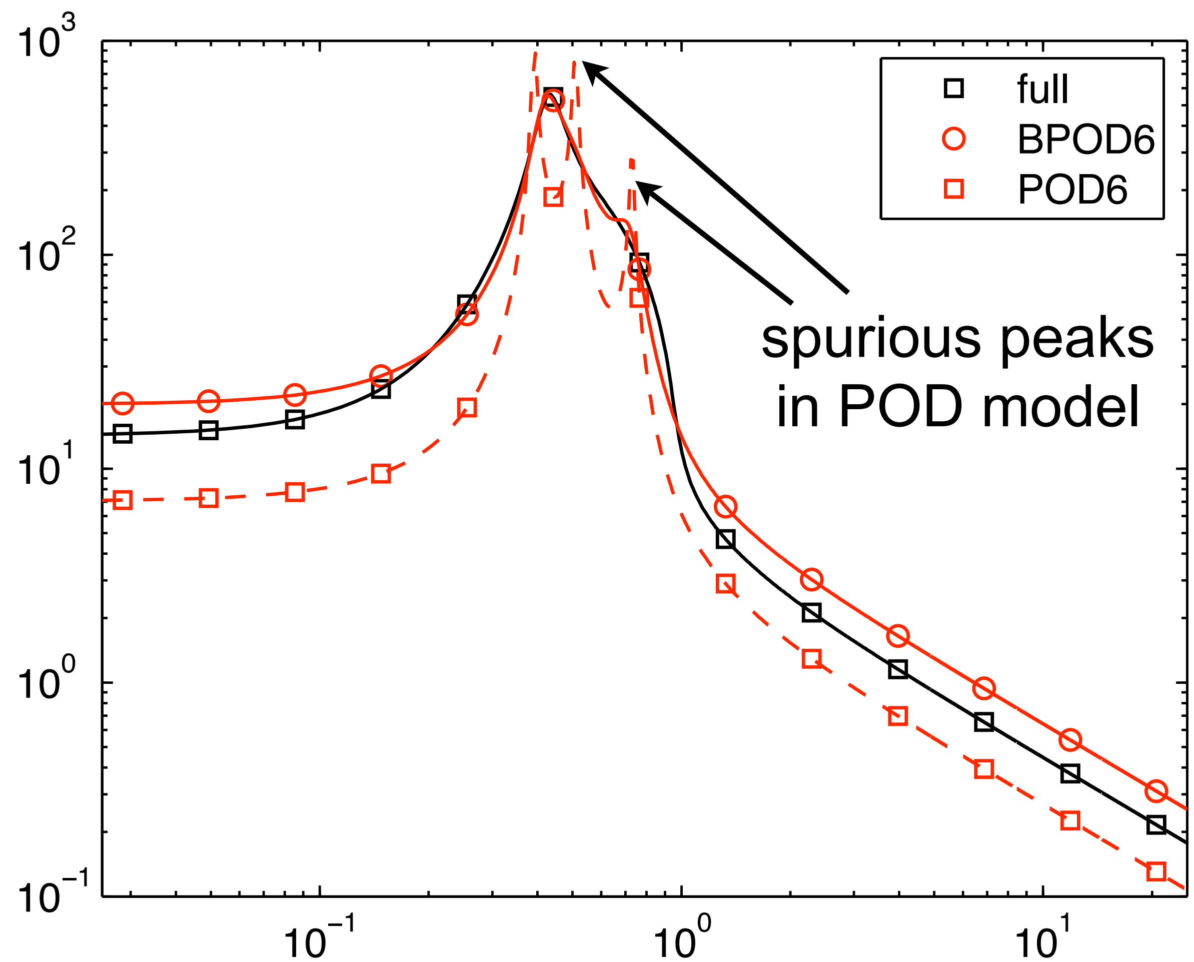


# Single wavenumber - frequency response

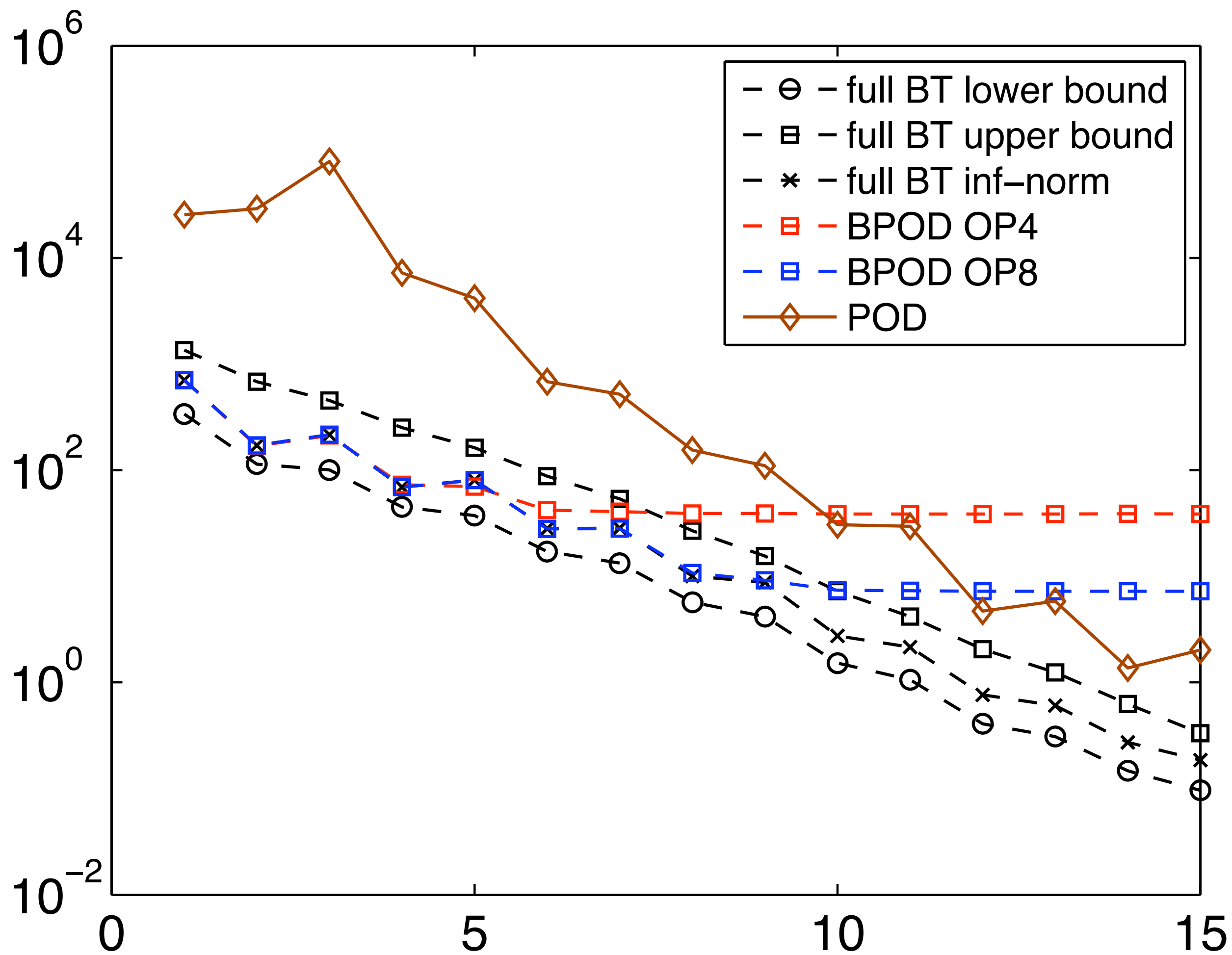
- For a single wavenumber, frequency response can be computed exactly
- BPOD captures the resonant peak even at low order
- POD slowly improves with additional modes, but has spurious peaks due to eigenvalues near the imaginary axis



# Single wavenumber - infinity norms

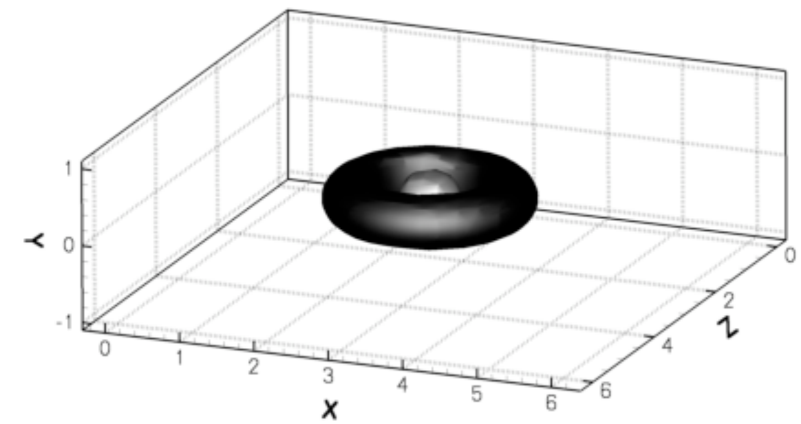
Infinity error norm bounds $\sigma_{r+1} \leq \|G - G_r\|_\infty \leq 2\Sigma_{j=r+1}^{n}\sigma_j$



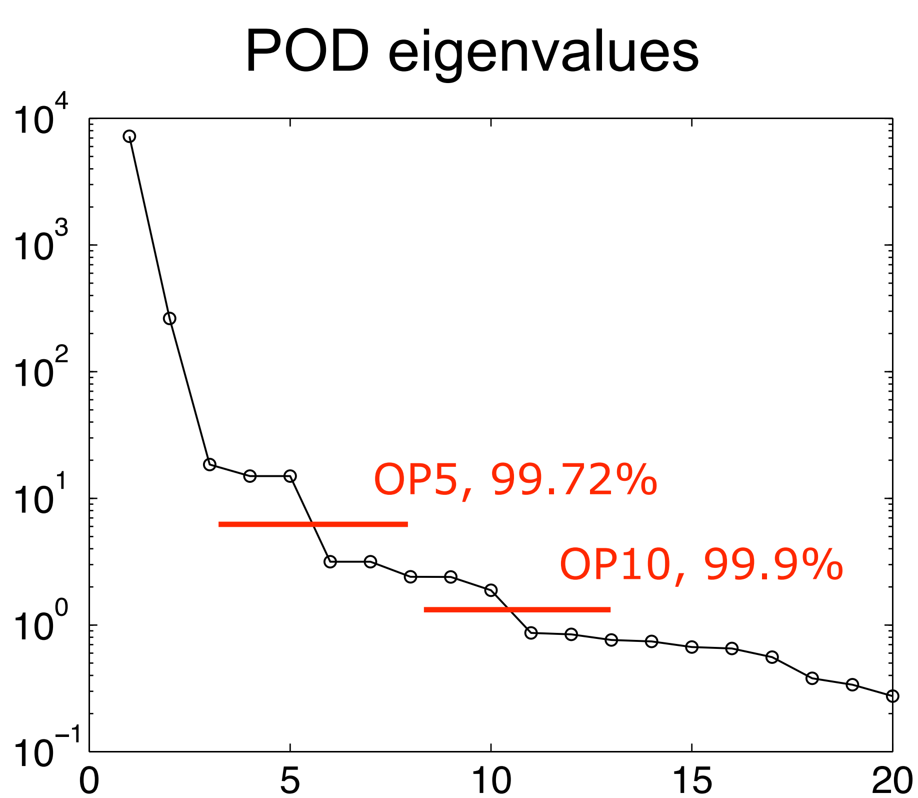

- Infinity norms of models also match those of exact BT up to approximately the rank of the output projection
- Again, POD 'catches up' only at a high rank

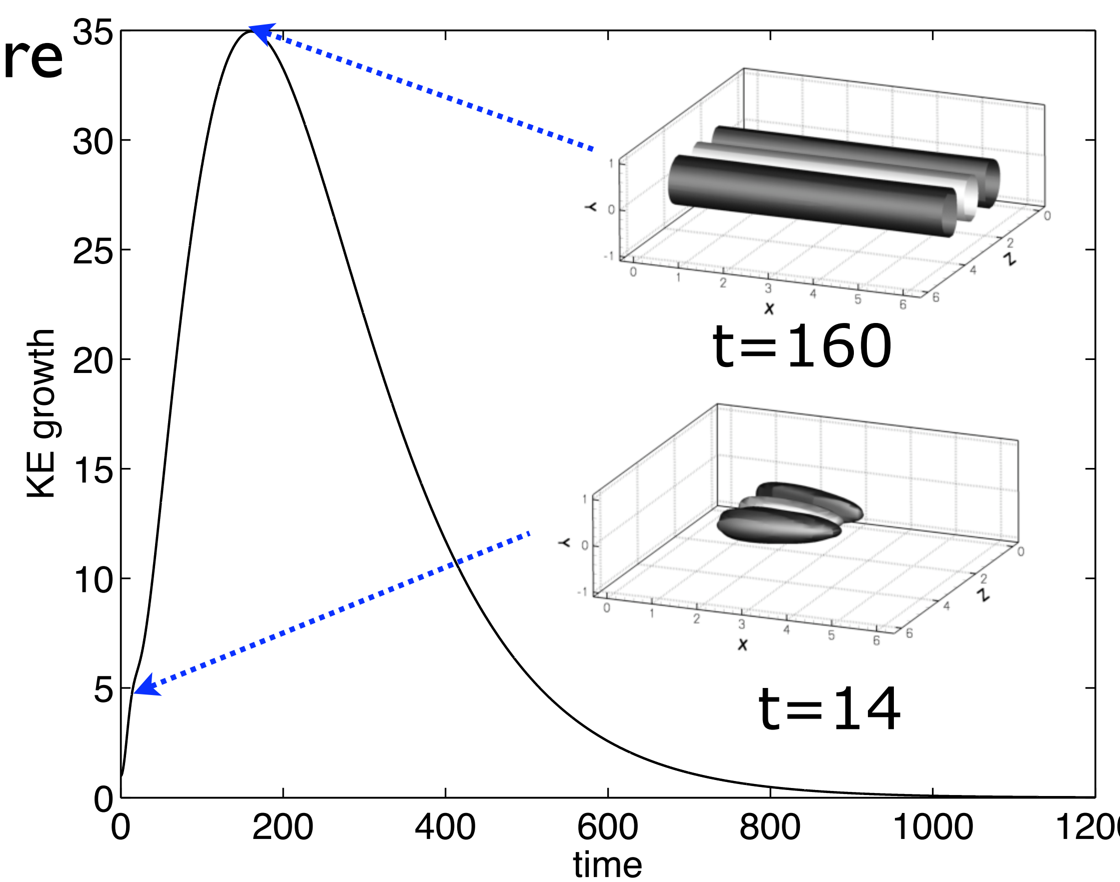
# Localized actuator/disturbance

- Periodic array of localized disturbances in center of channel
- Large system (32x65x32), 133,120 states, exact BT intractable
- Impulse response snapshots obtained via linearized DNS, Re=2000
- Complex initial transient develops into a streamwise-constant structure



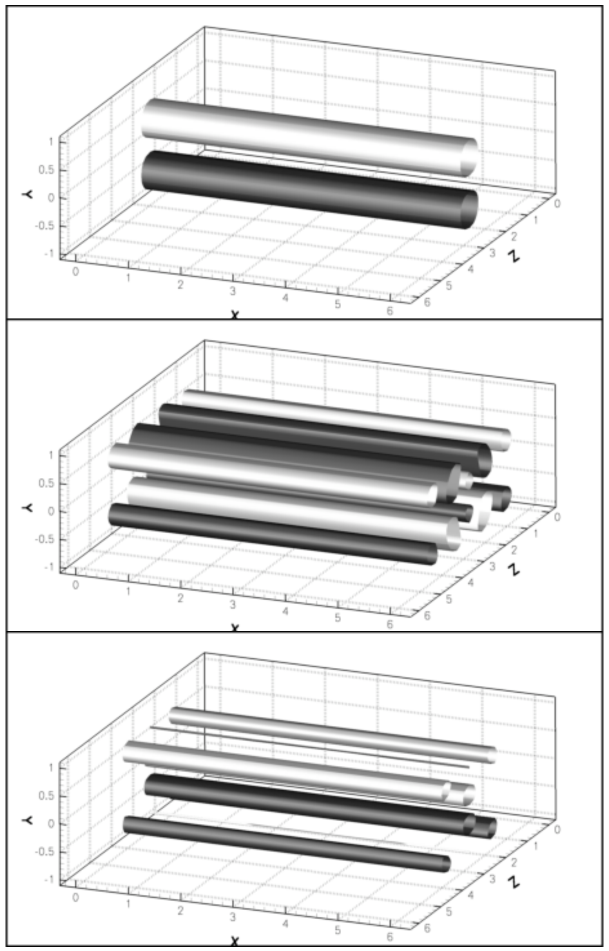

initial condition



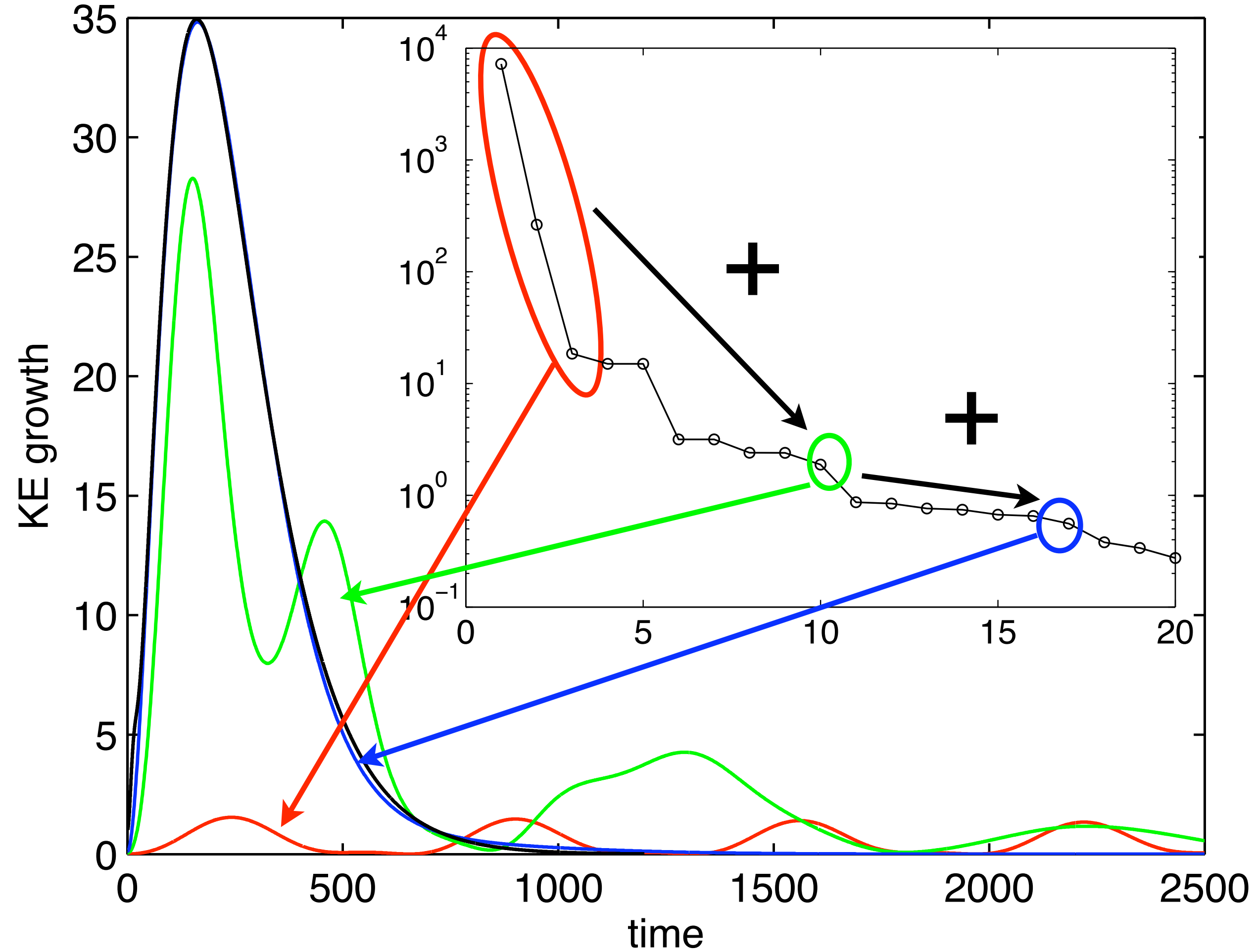



# Localized actuator - POD model performance

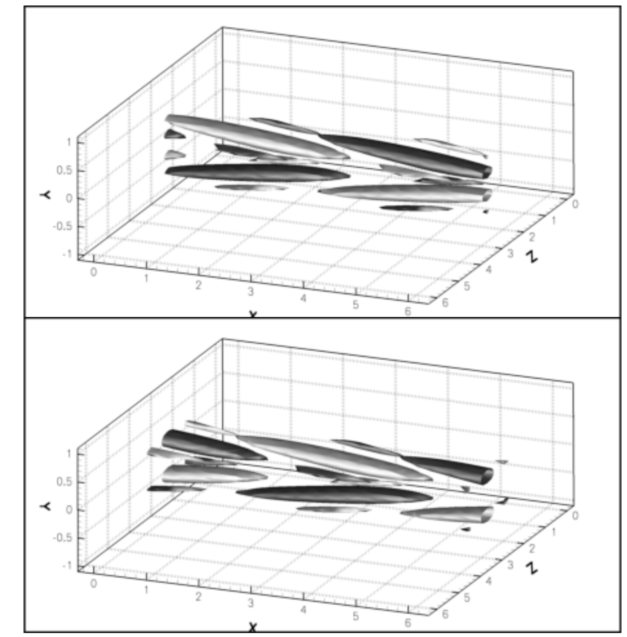
POD modes 1-3



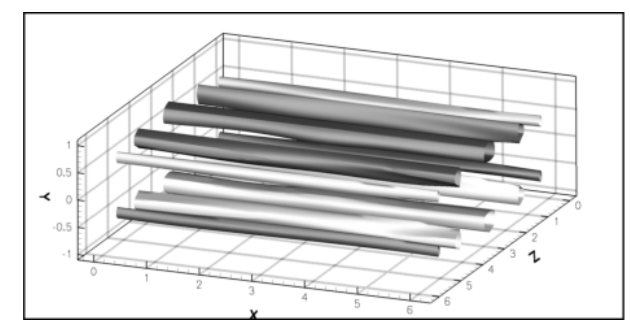
Standard POD



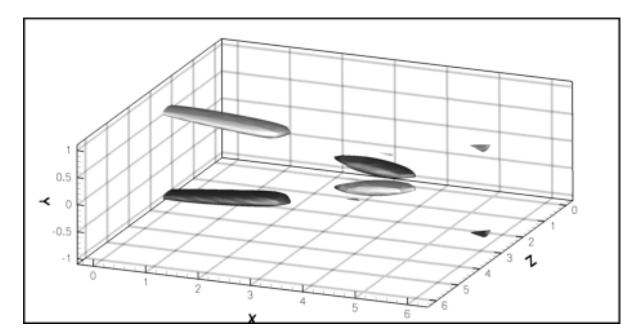

POD modes 4-5



POD mode 10



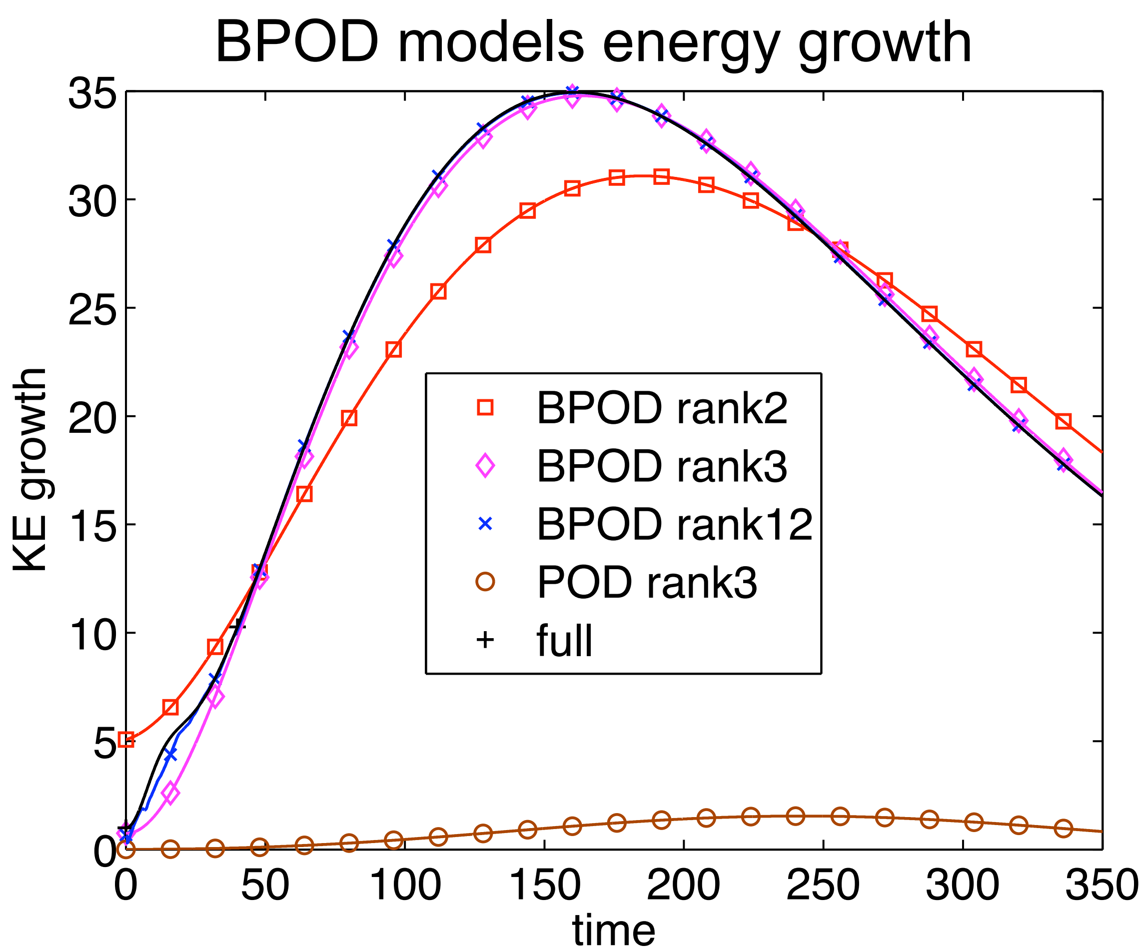
POD mode 17

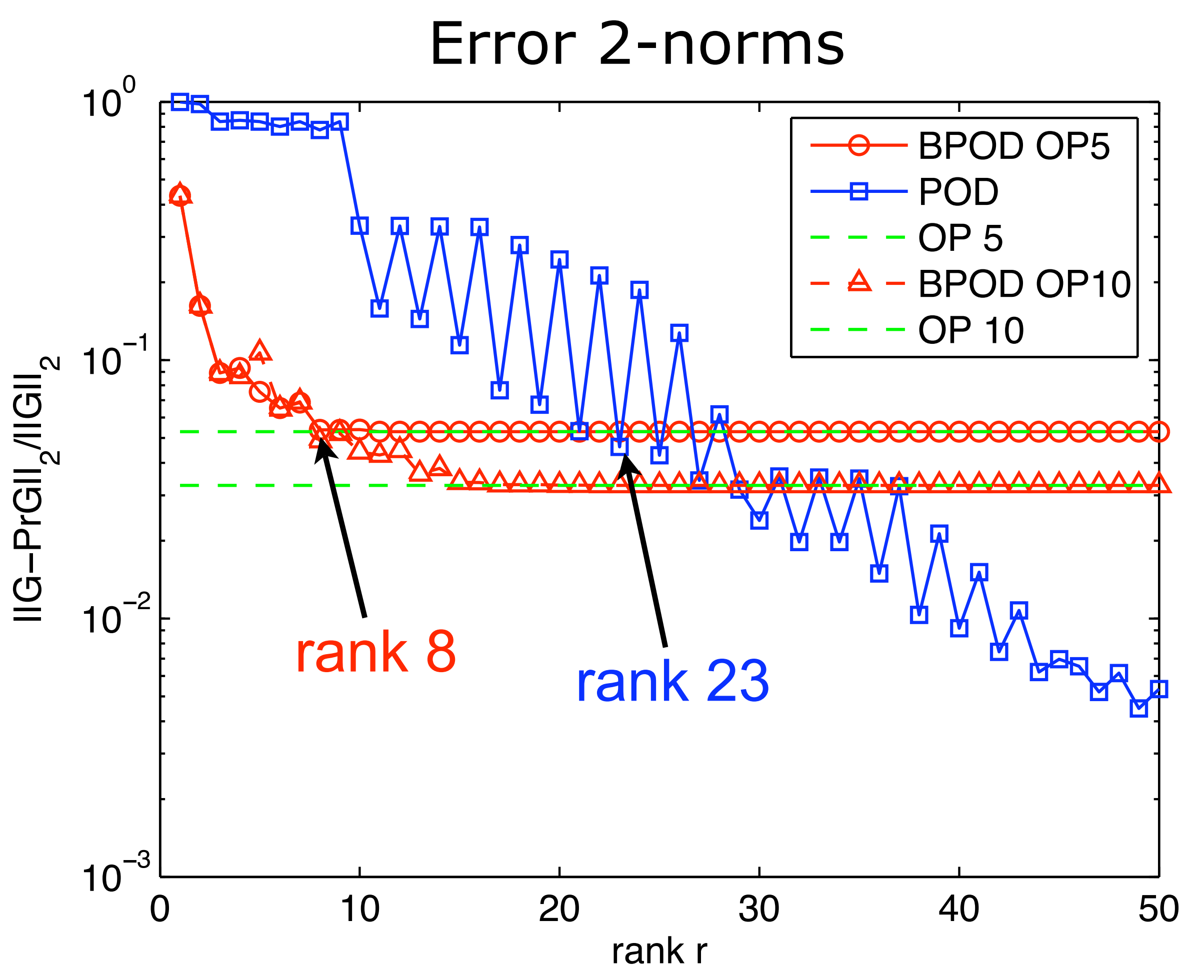


- 5-order model with modes 1,2,3,10,17 much better than 5-mode model with modes 1–5.

Conclusion: some low-energy POD modes are very important for the system dynamics. Can't naively use just the most energetic ones.

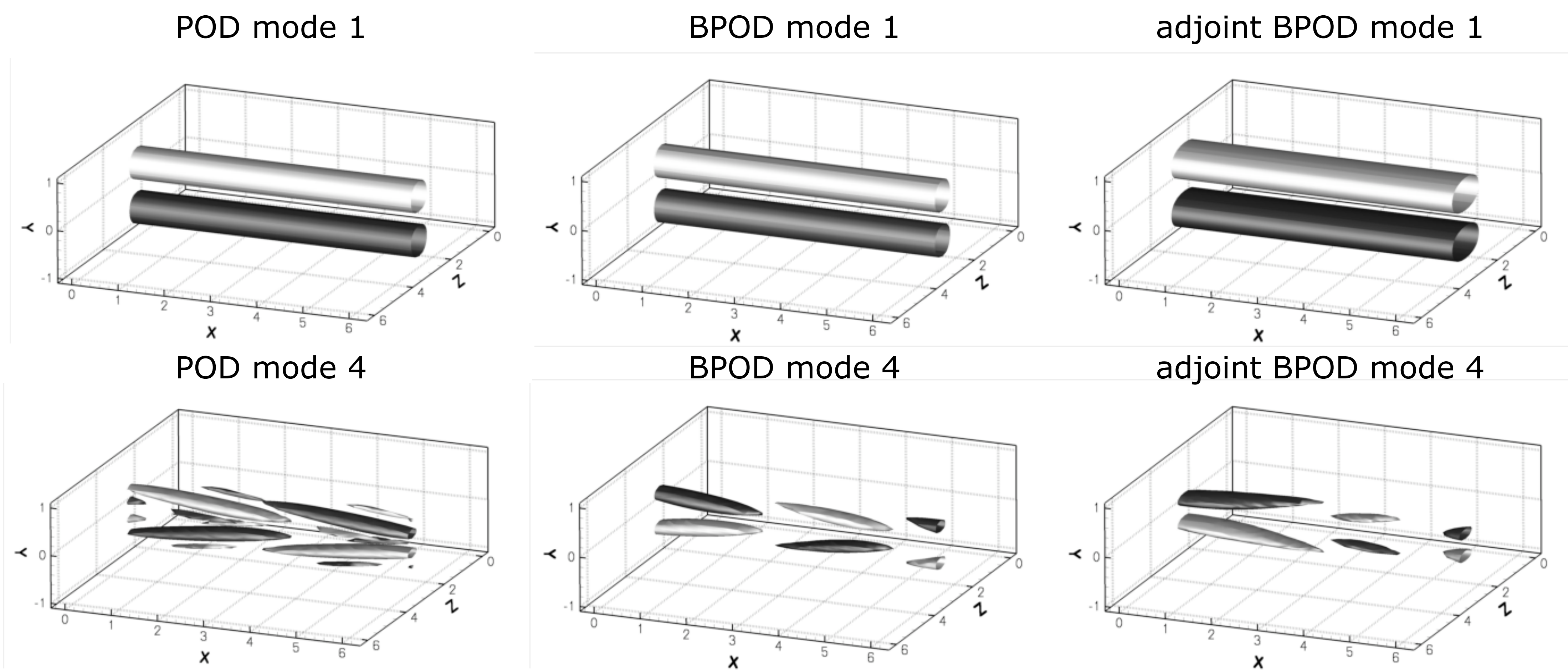
# Localized actuator - BPOD impulse response




- Three-mode BPOD model excellent at capturing the energy growth
- Rank 8 BPOD model sufficient to correctly capture the dynamics of the first five POD modes, compared to at least 23 POD modes
- Inclusion of some POD modes significantly deteriorates performance (splitting of the pairs of oscillating modes)

# Localized actuator - modes




BPOD and adjoint BPOD modes from OP5

Balancing modes and POD modes look similar but the adjoint modes are in general quite different => different dynamics of models

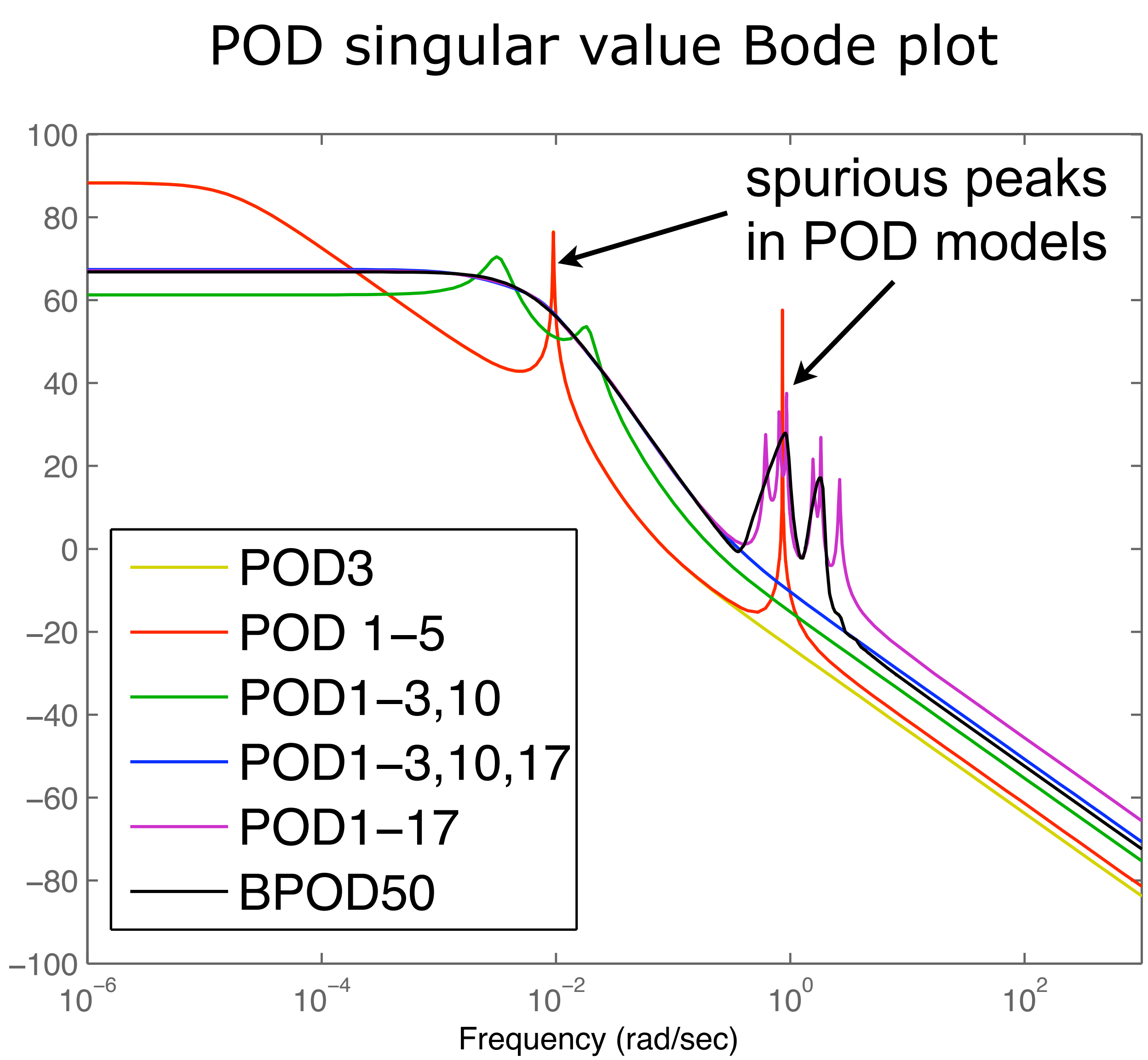
**POD**

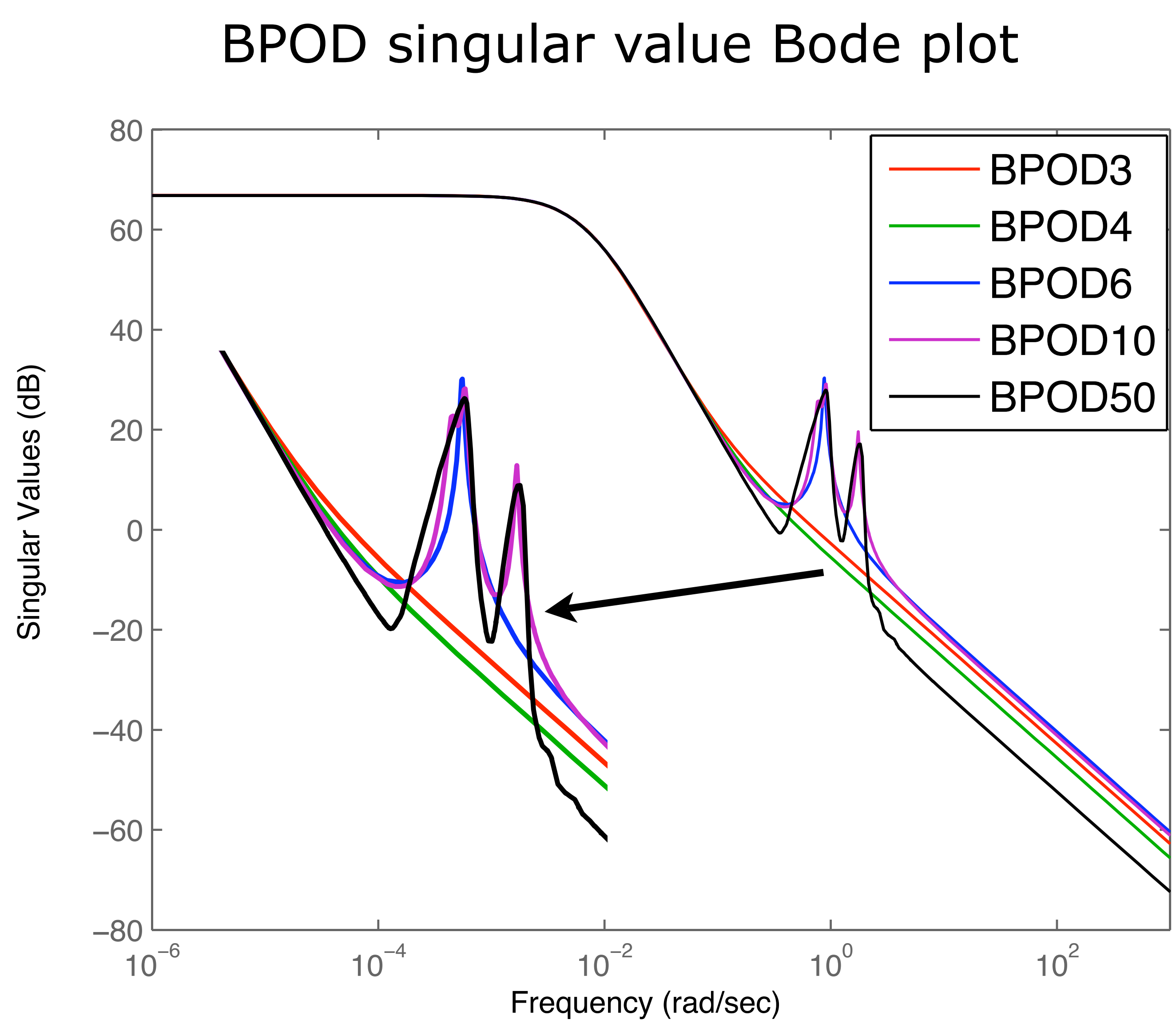
$$\dot{a}_j(t) = \langle \underline{\phi_j}, f(x) \rangle$$

**BPOD**

$$\dot{a}_j(t) = \langle \underline{\psi_j}, f(x) \rangle$$

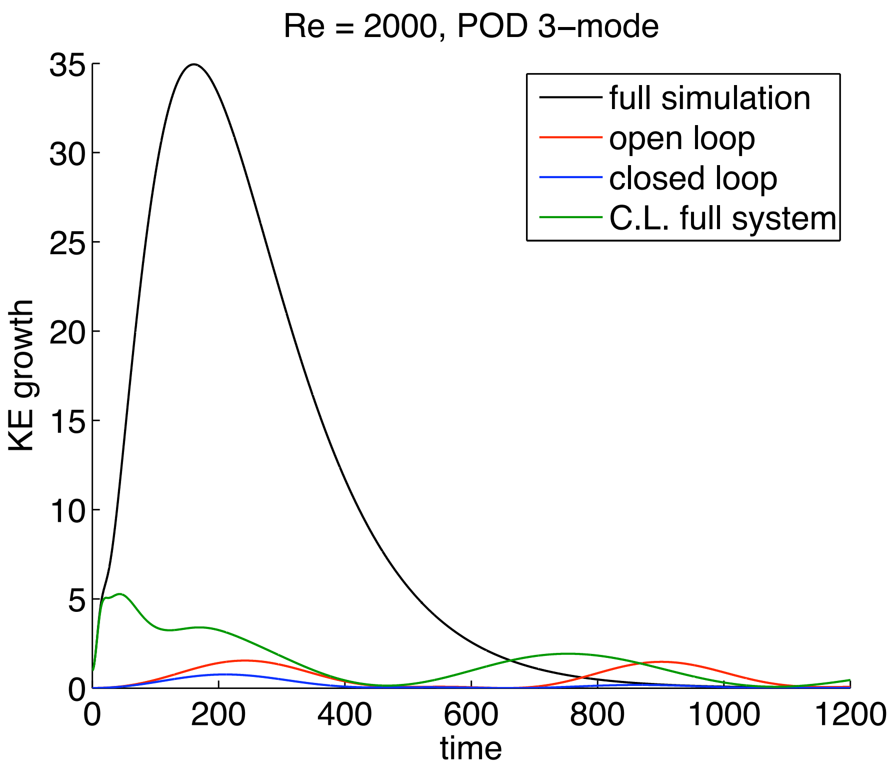
# Localized actuator - frequency response

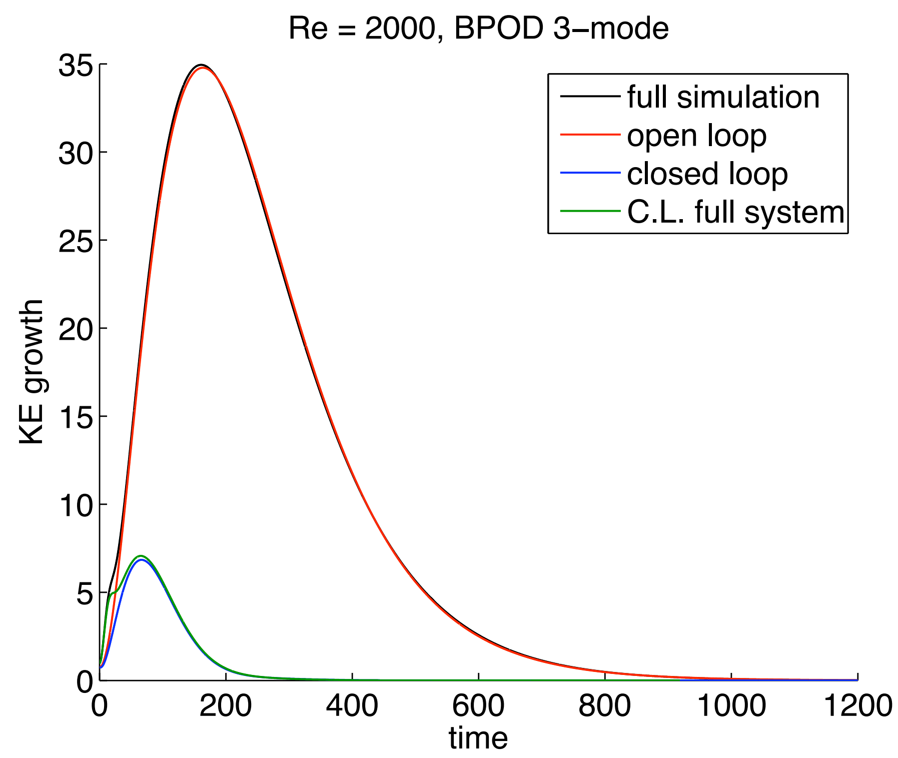


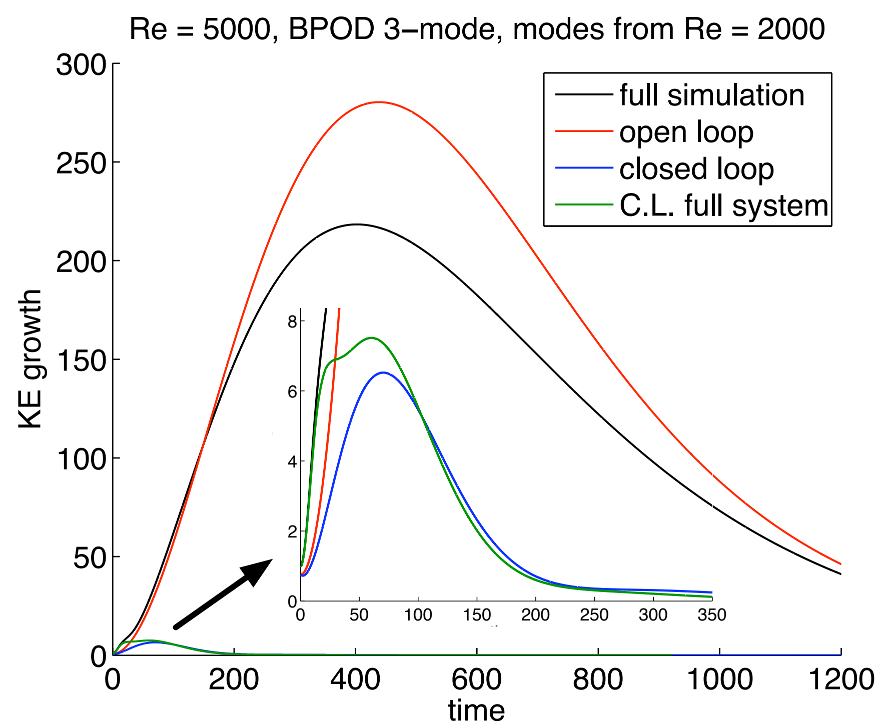

- BPOD 10-mode OP 50-mode model taken as 'full system'
- POD poorly captures low-pass behavior, spurious peaks
- Need pairs of BPOD modes to capture peaks

# Closed-loop control - localized actuator



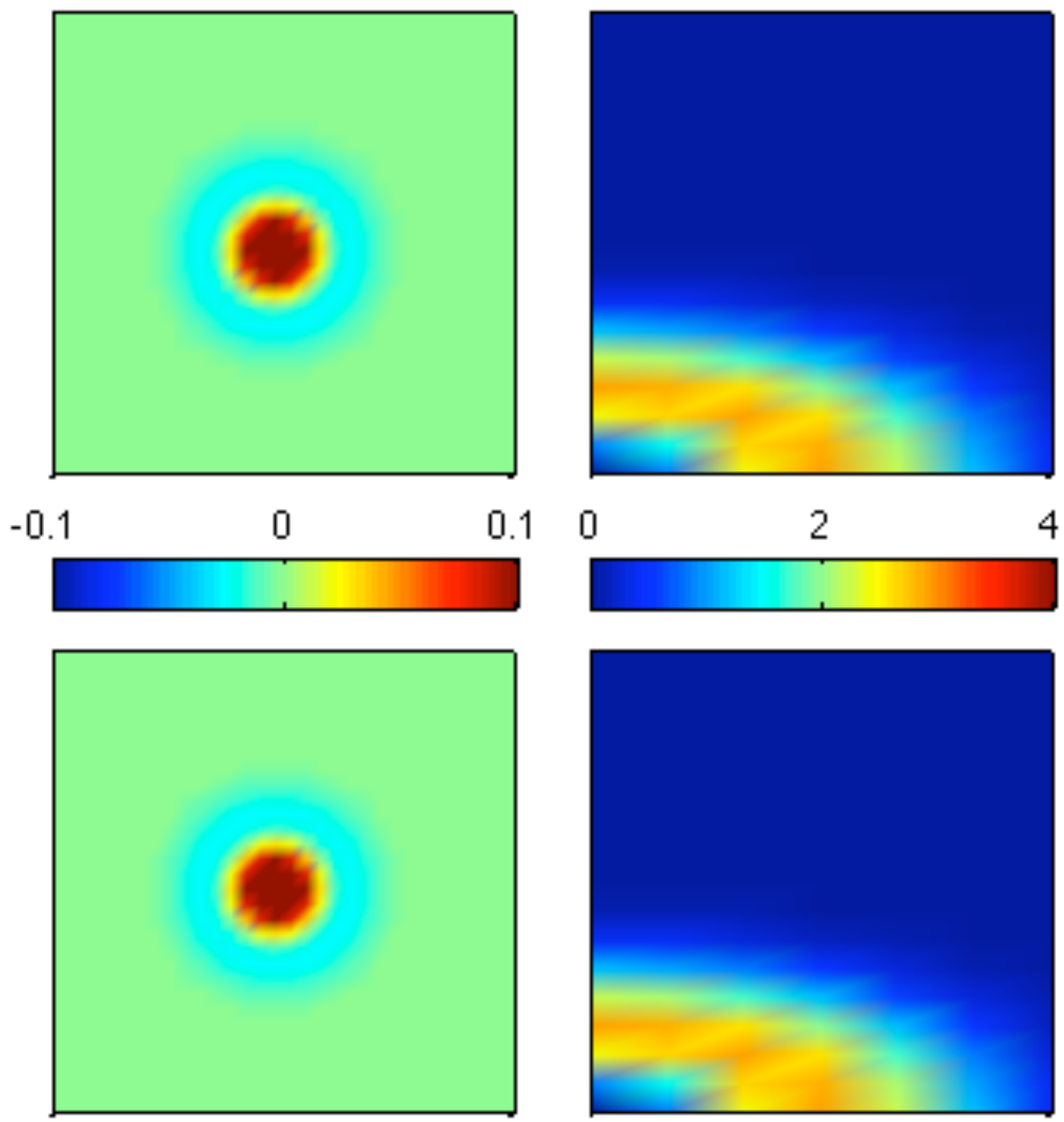




- Using the localized actuator to control a disturbance in channel center
- Standard LQR control design
- Using control gains from a 3-mode BPOD model reduces energy growth by a factor of 5



# Nonlinear Evolution of the Localized Perturbation

linear evolution of wall-normal velocity

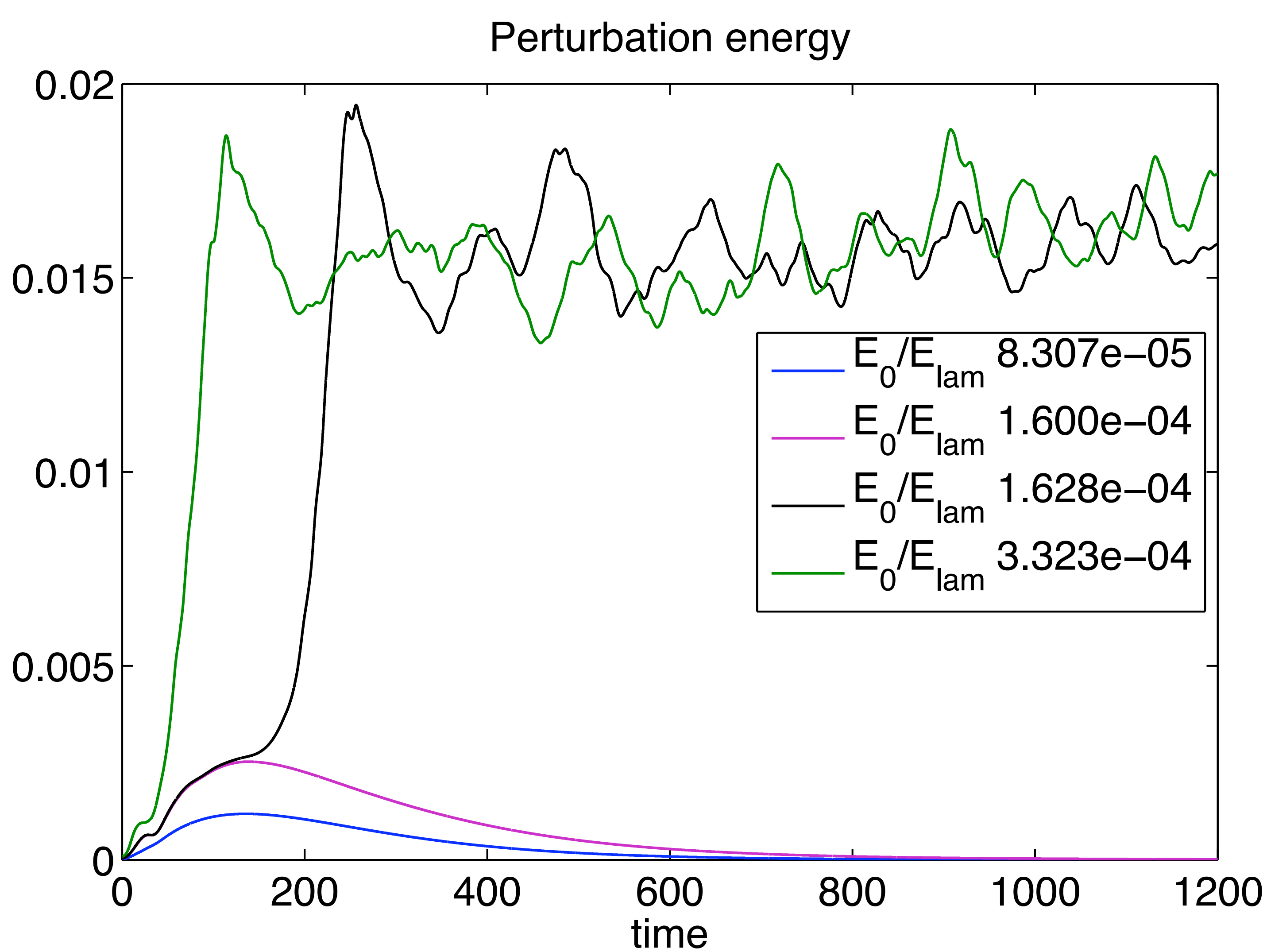



nonlinear evolution at $E_0/E_{lam} = 3.323 \times 10^{-4}$

- The spatial Fourier transform of the x,z plane at y=0 illustrates the perturbation evolution
- In the linear case the wavenumbers decay independently after the large transient growth
- $E_{lam} = 0.2667$ is the energy density of the mean laminar flow
- Transition for very small values of initial energy $E_0$
- The so-called β-cascade [Henningson et al, 1993] is observed in the nonlinear evolution - higher spanwise wavenumbers are introduced rapidly

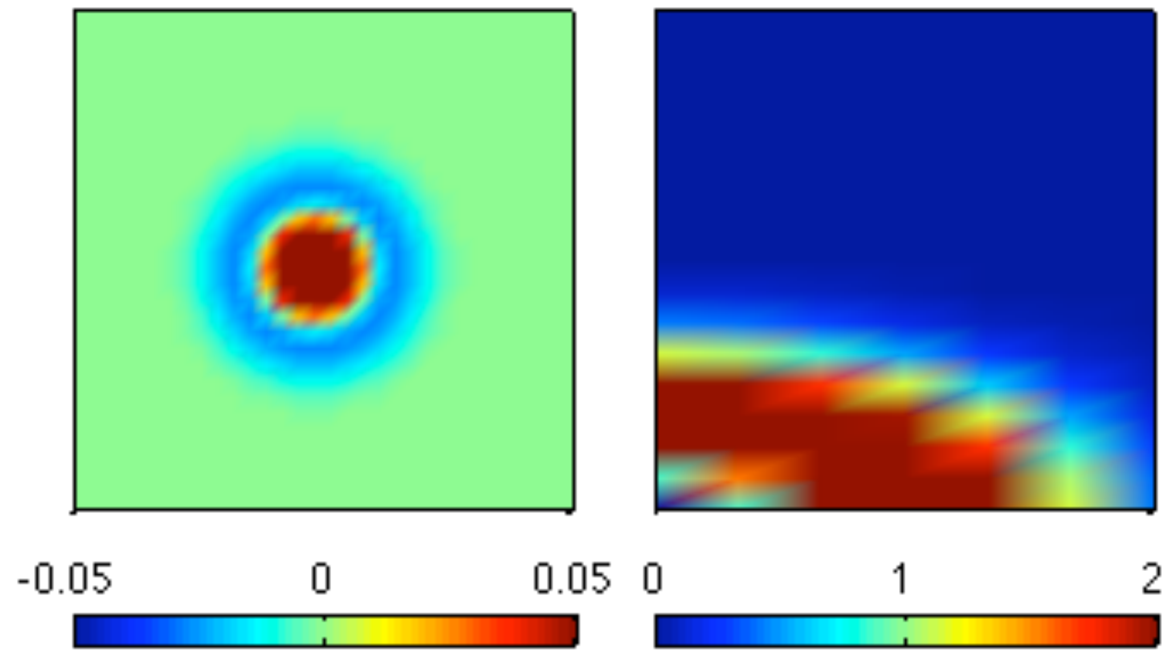
# Delaying Transition Using Feedback Control

- Try to increase the transition threshold of a localized perturbation (after Reddy et al)
- The threshold is defined as the energy density of the initial perturbation above which the flow transitions to turbulence
- Threshold found to be at $E_0 = 1.614 \times 10^{-4}$ of the mean flow energy of the laminar profile, $E_{lam} = 0.2667$



# Closed-Loop Control

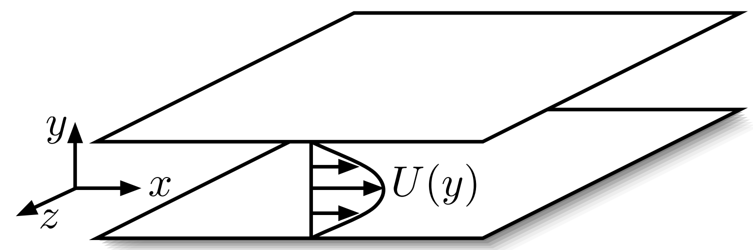
- The feedback gains computed using LQR for the linear system are used in a full nonlinear simulation with $E_0/E_{lam} = 3.323 \times 10^{-4}$
- An 'aggressive' controller (R=0.1 in LQR) manages to suppress the disturbance

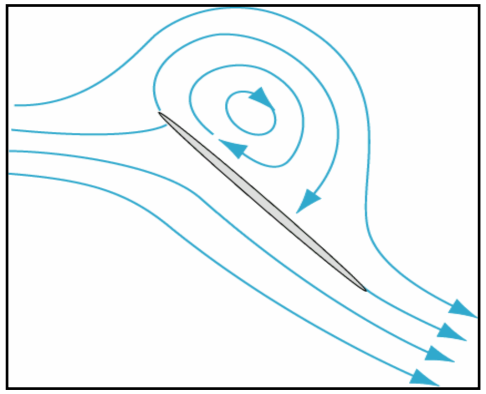


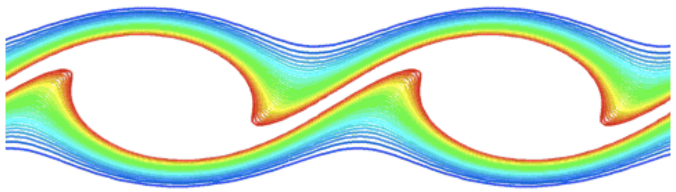

- Explanation: the BPOD modes do not have components at high β, and are not able to suppress high betas once they arise, but the 'aggressive' controller suppresses low β wavenumbers so that the higher β's emerge at very low amplitudes and decay linearly
- Transition threshold increased by a factor of 17 for R=0.01
- Work in progress: see how projection of full N-S equations onto linear BPOD modes will model the perturbation evolution, and possibly design a nonlinear controller

# Outline

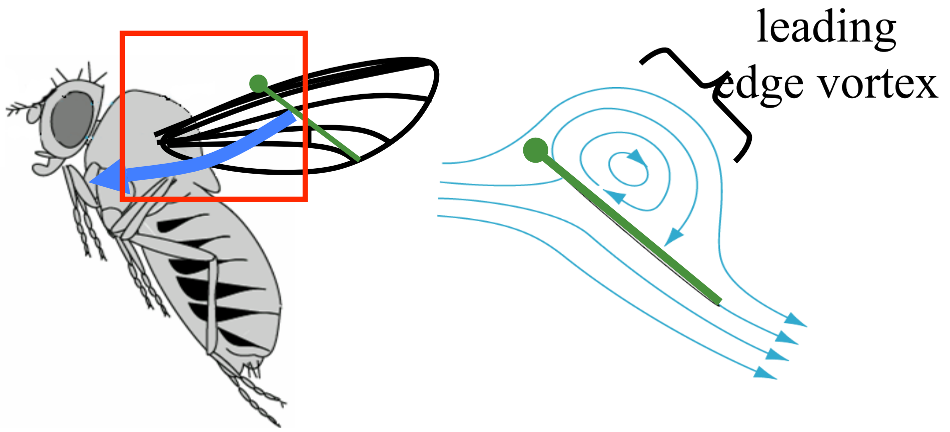
- Reduced-order models: POD and balanced truncation
  - Importance of inner product for Galerkin projection
  - Balanced truncation
  - Method of snapshots
- **Applications**
  - Linearized channel flow
  - **Separating flow past an airfoil**
- Dynamically scaling POD modes
  - Free shear layer
  - Scaled basis functions
  - Template fitting
  - Equations for the shear layer thickness



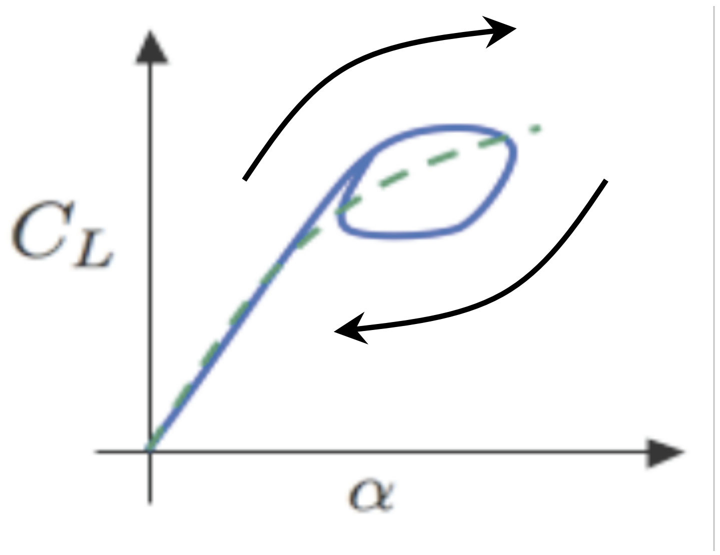


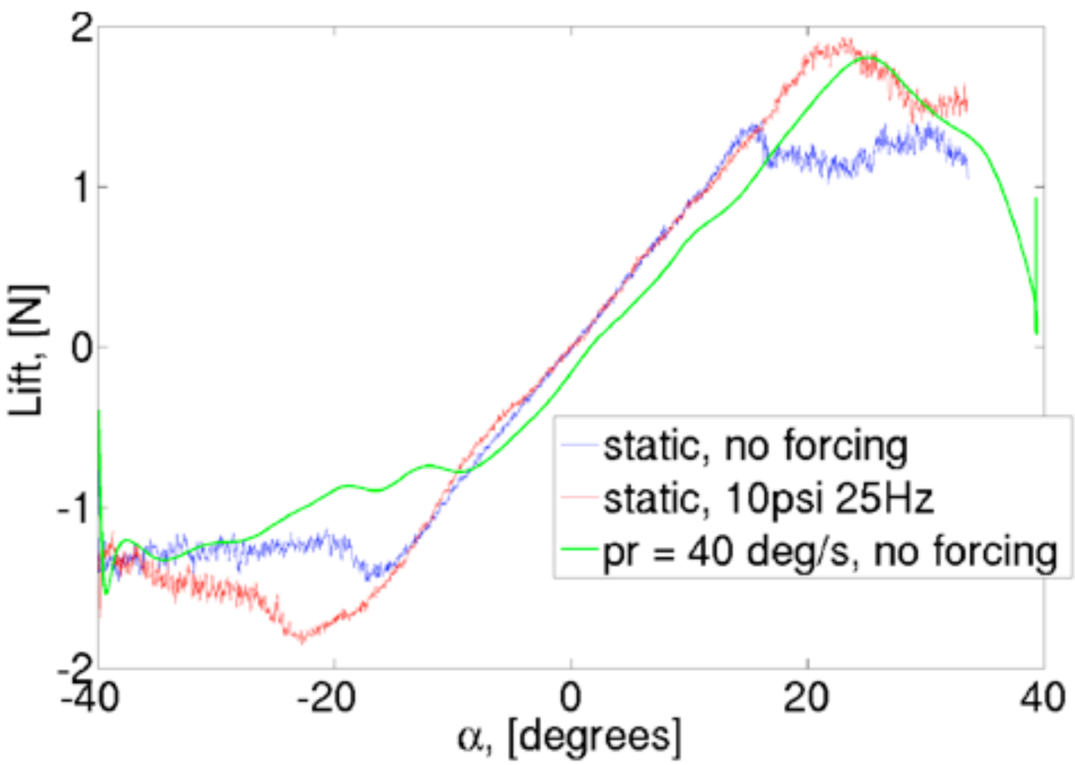
# Motivation




- Leading edge vortices can provide high lift
- MURI with Caltech (Colonius), IIT (Williams), and Northeastern (Tadmor)
- Goal: Stabilize these LEVs using feedback control
- High transient lift in pitching airfoils due to dynamic stall vortex

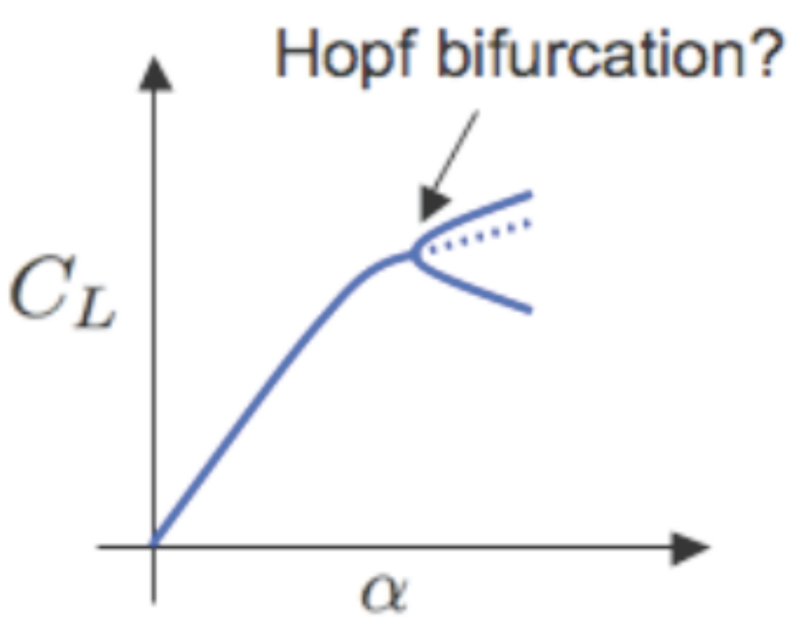
Pitching airfoil



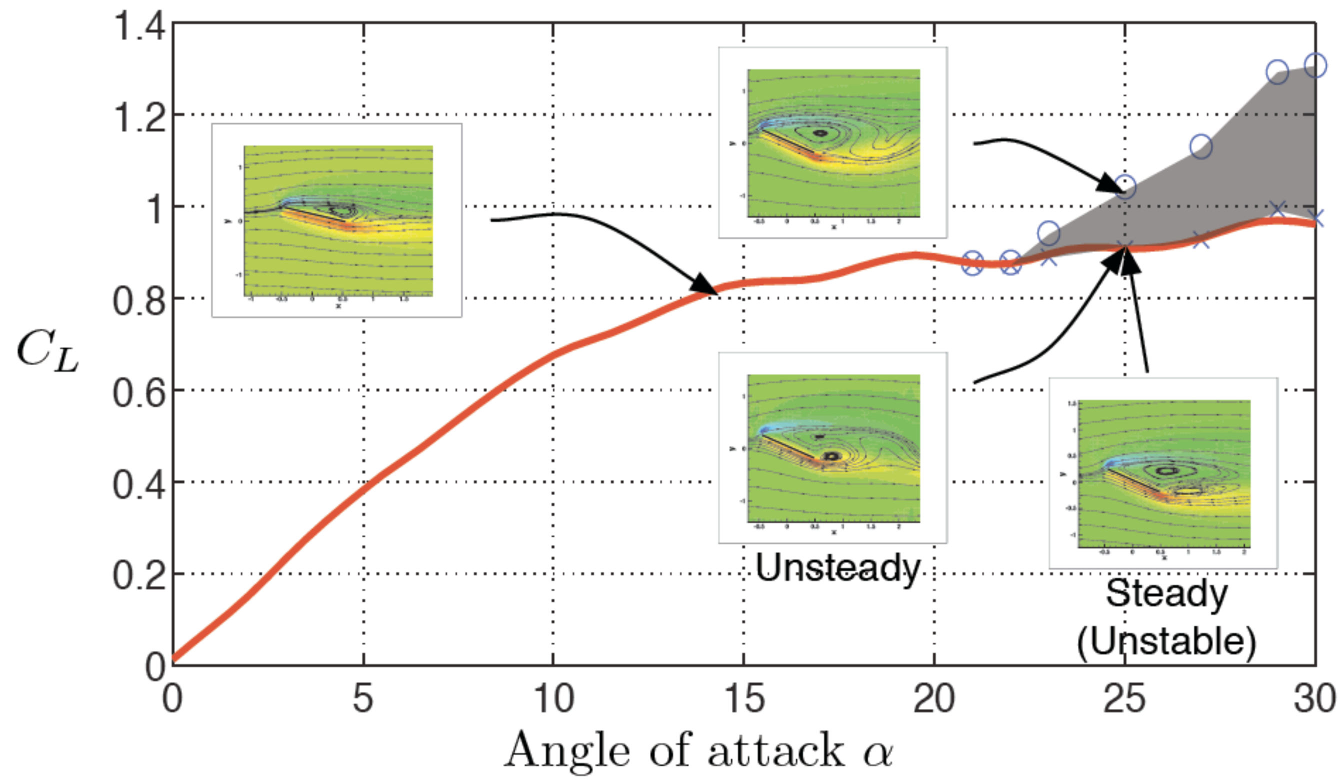



# Dynamical behavior

- With increasing AoA, flow undergoes a Hopf bifurcation
- Reduced order models to stabilize unstable steady states at high AoAs




$C_L$ at steady state



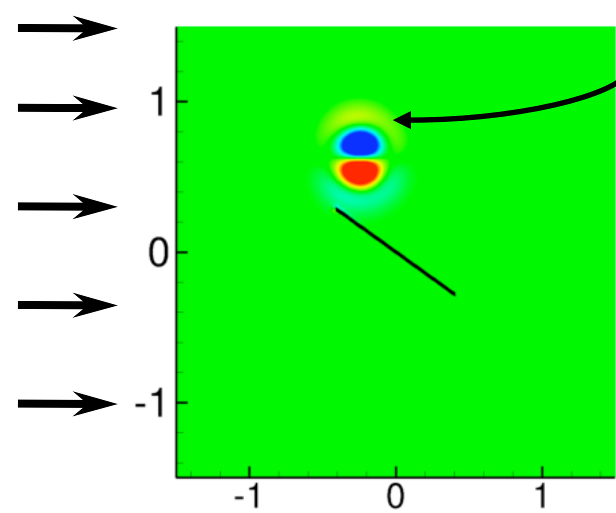

Are there high-lift unstable steady states in low aspect ratio airfoils?

# Model problem

Actuator: localized body force

***Re* = 100**

**AoA = 25 or 35**




- A fast null-space based immersed boundary scheme for numerical simulations

(T. Colonius and K. Taira, CMAME, 2007)

## Steady state analysis

- Compute steady states using a wrapper around the DNS

DNS

$$u^k \longrightarrow \boxed{\Phi_T(u^k;\mu)} \longrightarrow u^{k+T}$$

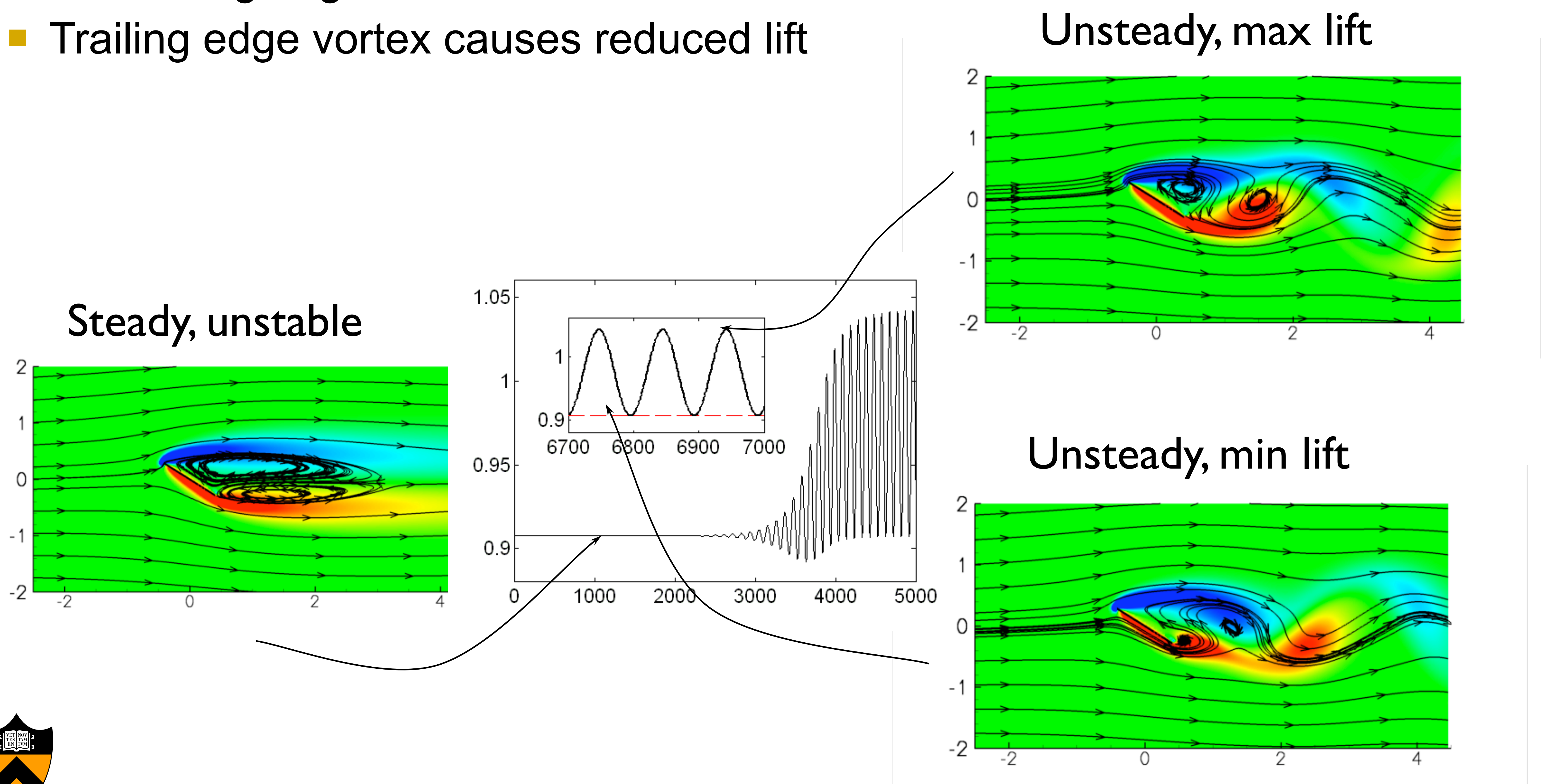
Define: $g(u) = u^T - u$

- Solve for zeroes of *g(u)* using Newton-GMRES

Barkley and Tuckerman,'99, Kelley, Kevrekidis, and Qiao,'02, Ahuja et al., '07

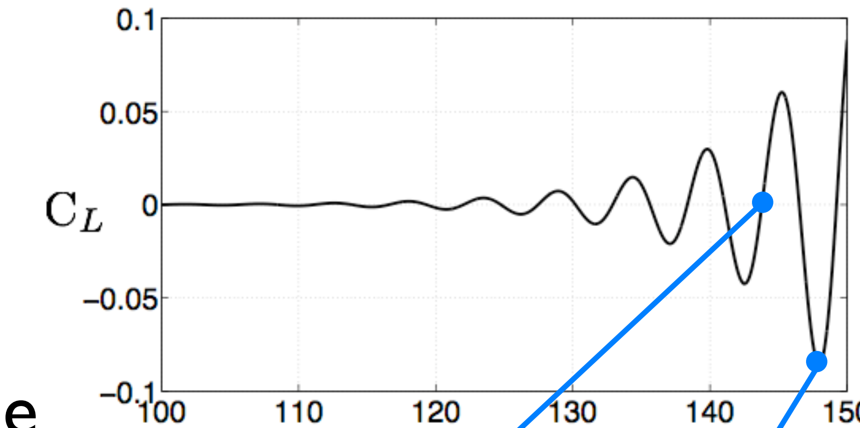
# Unstable steady state, AoA = 35

- Steady state lift close to the min. lift of the unsteady case
- No leading edge vortex
- Trailing edge vortex causes reduced lift

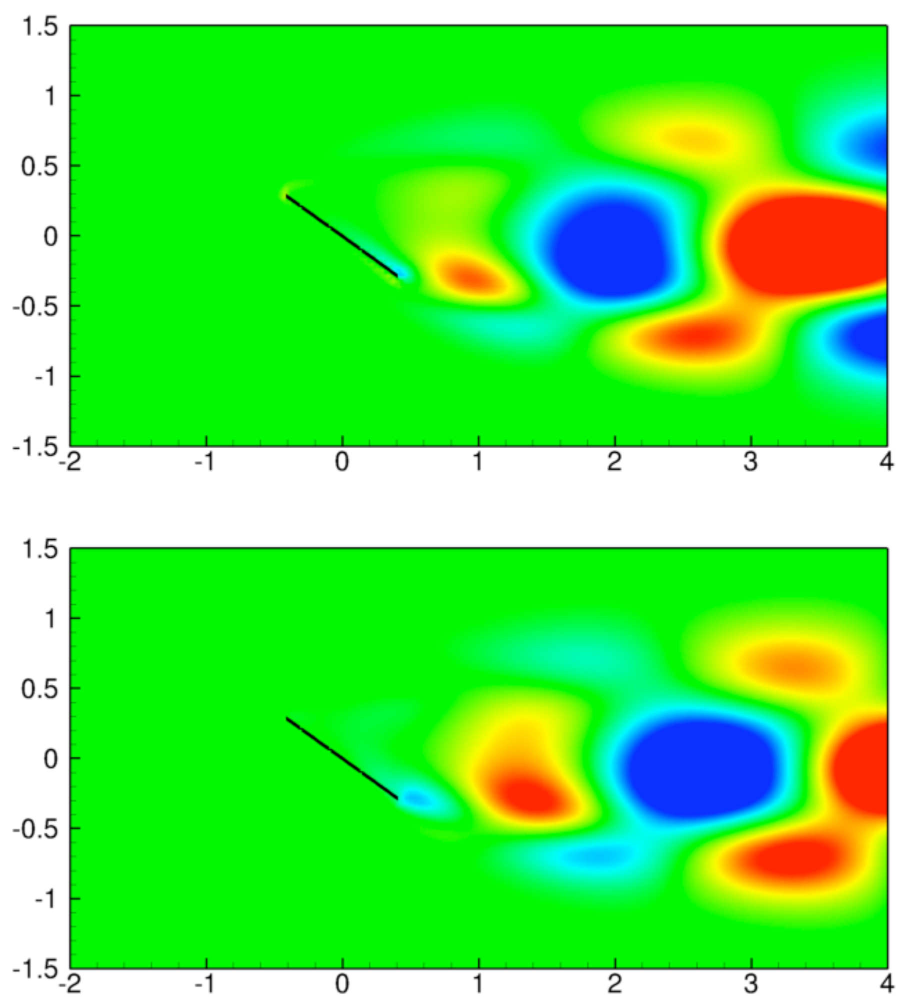


# Linear stability analysis

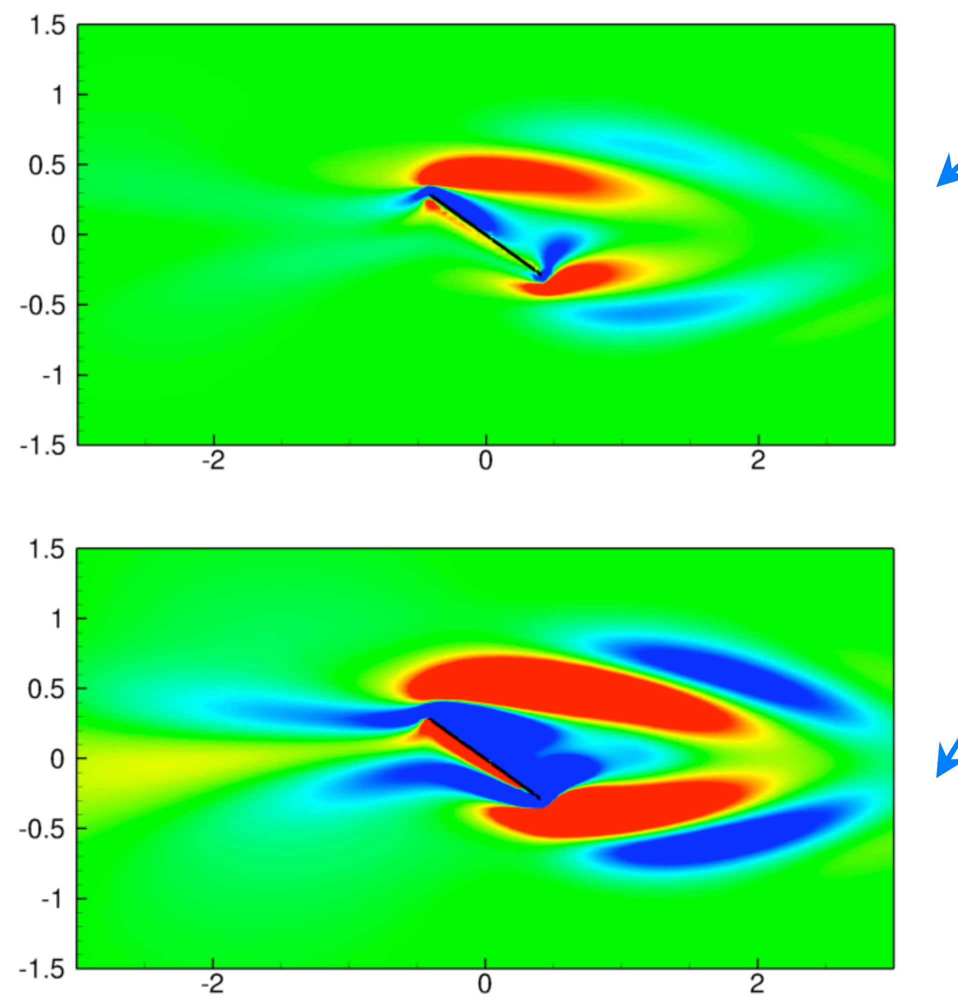
- Find the basis spanning the unstable eigenspace of the linearized and adjoint flows
- Run the linear simulations with a zero initial condition + $10^{-8}$ random noise




Right eigen-space




Left eigen-space

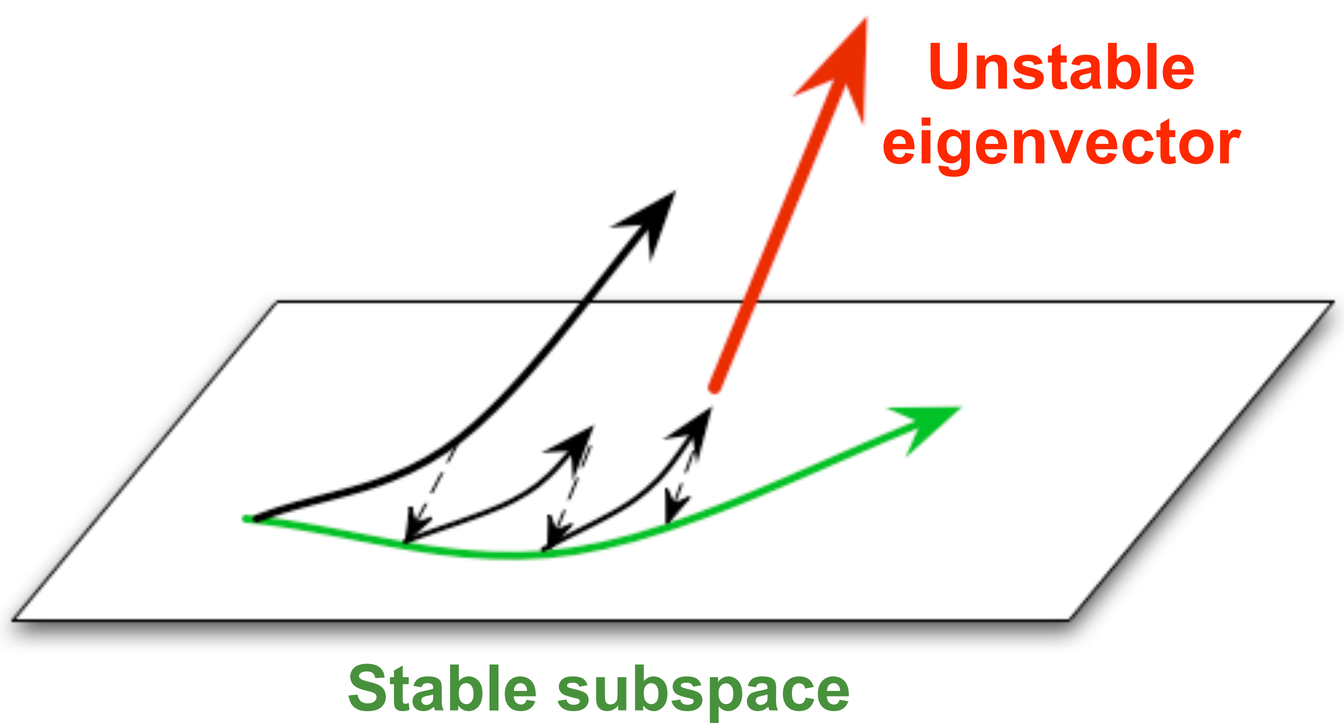


# Reduced-order models for unstable systems

- Decouple stable and unstable subspaces
- Obtain balancing transformation for the stable subspace

$$\frac{d}{dt}\begin{pmatrix} x_s \\ x_u \end{pmatrix} = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix}\begin{pmatrix} x_s \\ x_u \end{pmatrix} + \begin{pmatrix} B_s \\ B_u \end{pmatrix} u$$

- Snapshot based procedure: project out the unstable component at each time step




Balanced truncation for unstable systems, Zhou et al., '99

# Model reduction: unstable system

**Linearized NS eqns, $10^5$**

$$u \longrightarrow \boxed{\frac{d}{dt}\begin{pmatrix} x_s \\ x_u \end{pmatrix} = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix}\begin{pmatrix} x_s \\ x_u \end{pmatrix} + \begin{pmatrix} B_s \\ B_u \end{pmatrix} u} \longrightarrow \boxed{y = \begin{pmatrix} \Theta^T x_s \\ x_u \end{pmatrix}} \longrightarrow$$

$y_s$ = proj. of $x_s$ onto POD modes

$x_u$ = unstable state

$$\mathrm{x}_s = \Phi a_s$$

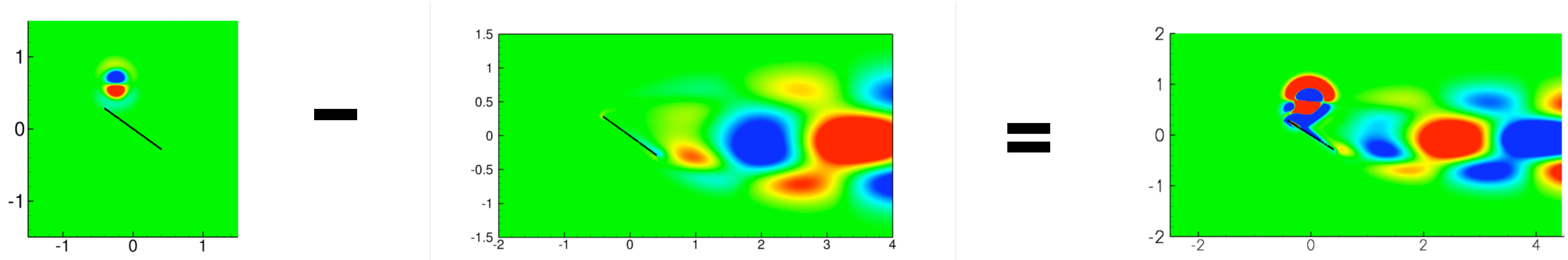
$$\Psi^T \Phi = I$$

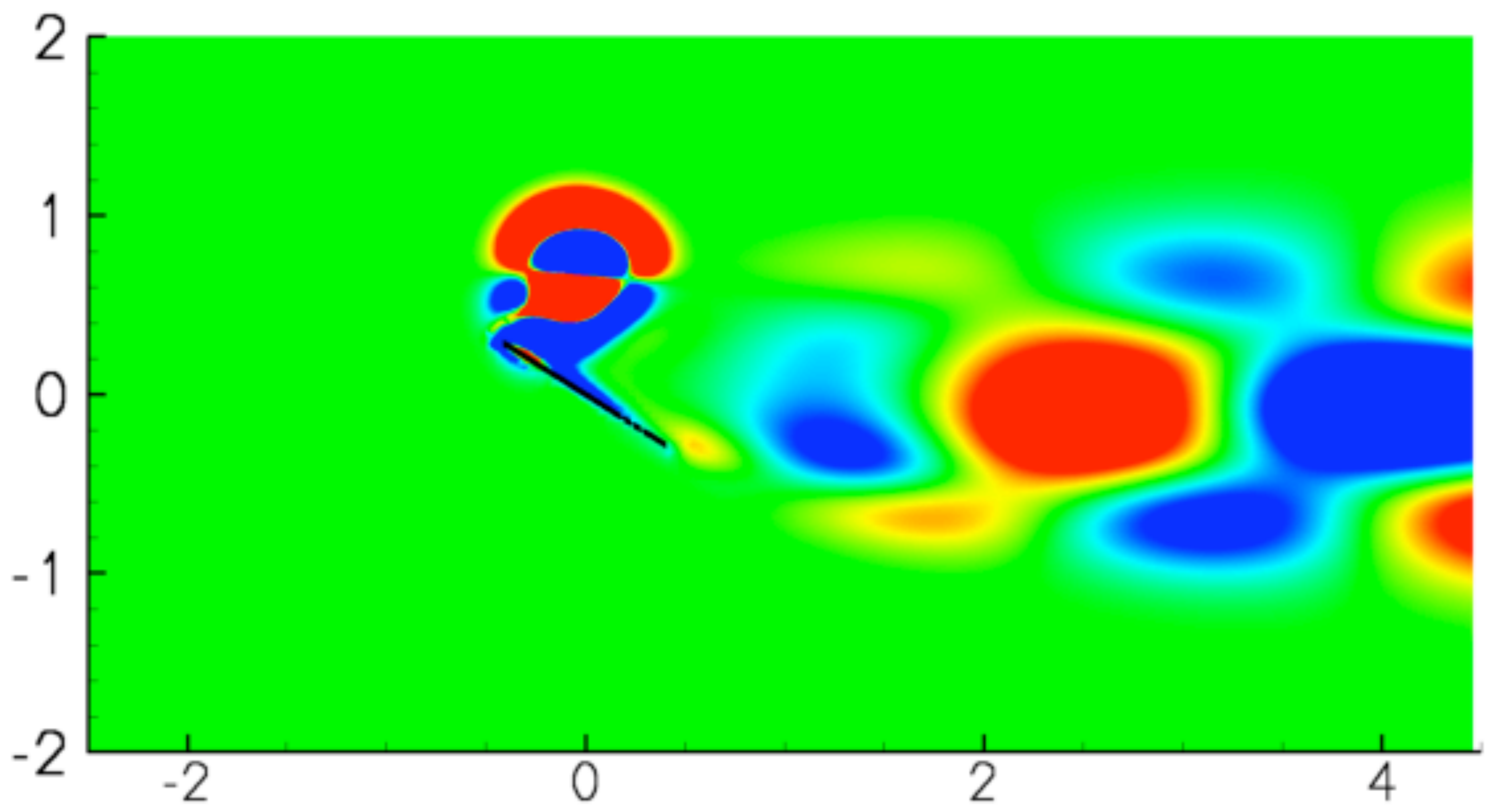
**Reduced order model, 10-50 eqns.**

$$u \longrightarrow \boxed{\frac{d}{dt}\begin{pmatrix} a_s \\ x_u \end{pmatrix} = \begin{pmatrix} \Psi^T A_s \Phi & 0 \\ 0 & A_u \end{pmatrix}\begin{pmatrix} a_s \\ x_u \end{pmatrix} + \begin{pmatrix} \Psi^T B_s \\ B_u \end{pmatrix} u} \longrightarrow \begin{pmatrix} y_s \\ x_u \end{pmatrix}$$

# Impulse response: stable subspace

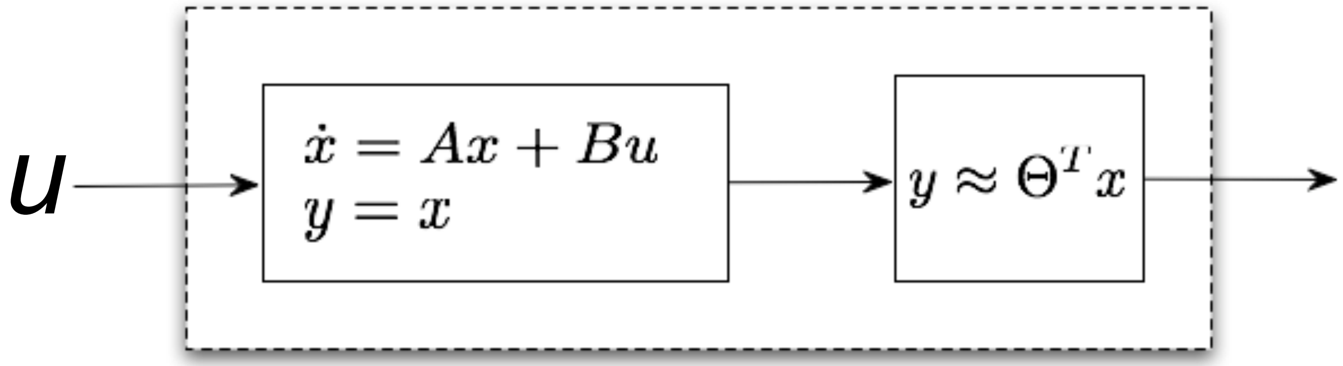
- Project out the unstable component from the initial condition



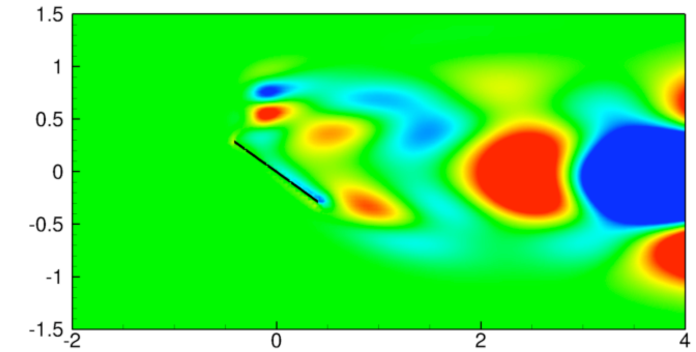

Vorticity contours:
Positive in red and
negative in blue

# Adjoint impulse response

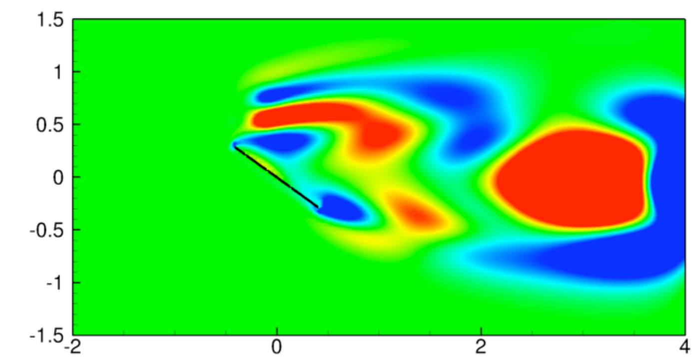



Projection onto POD modes

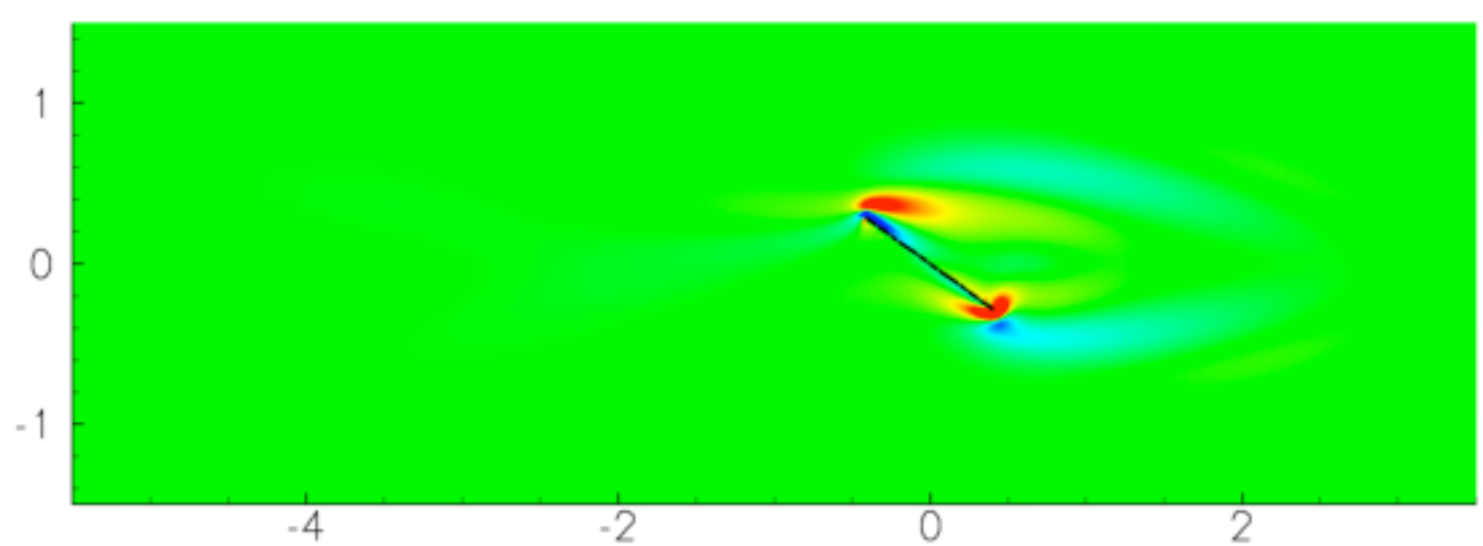
Mode 1



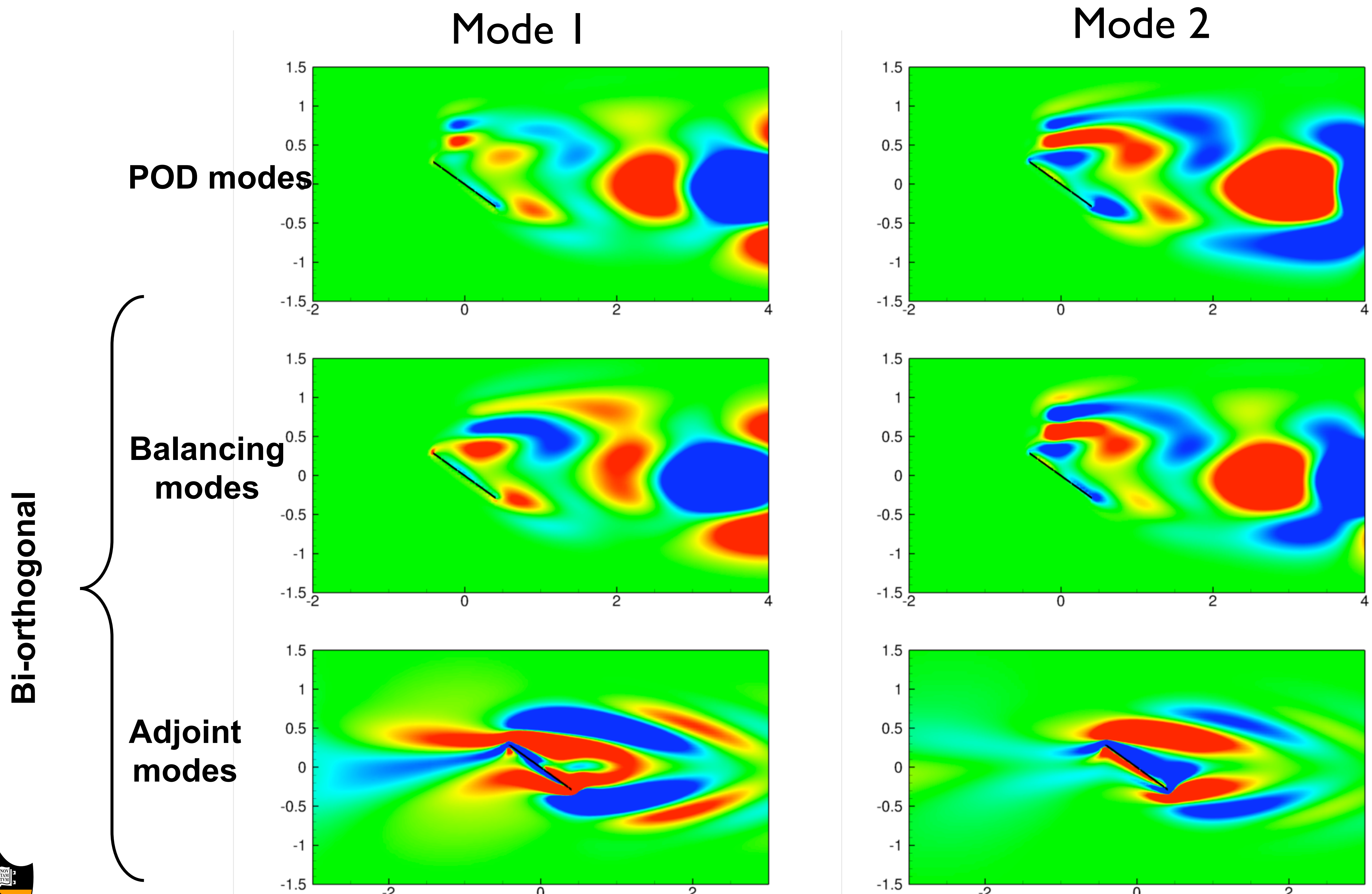
Mode 2



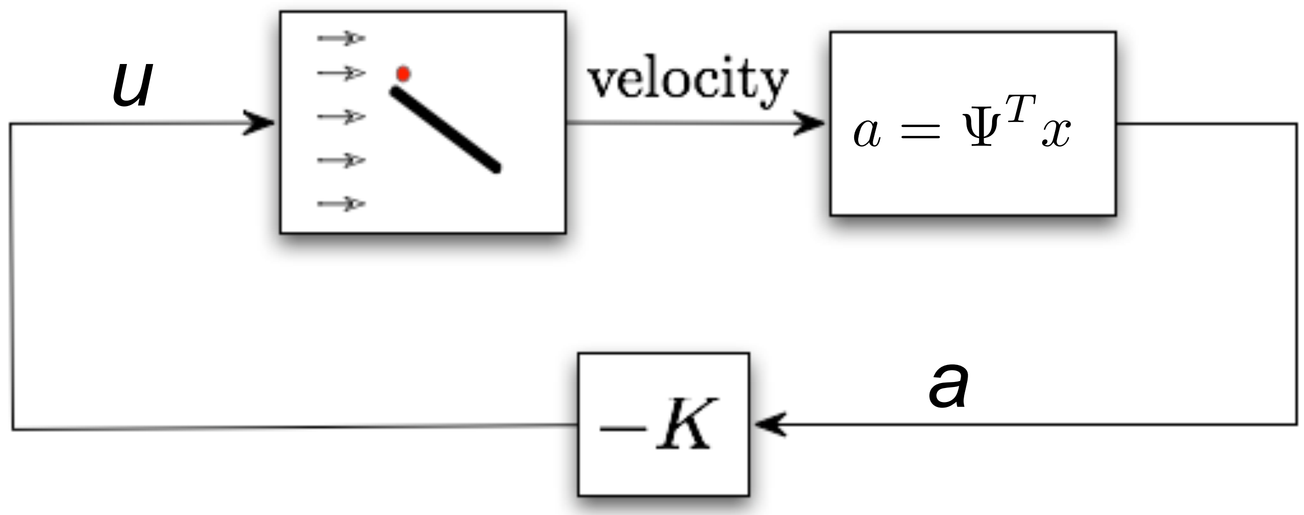
- Four POD modes capture 95% energy
- Adjoint solves with these POD modes as initial conditions



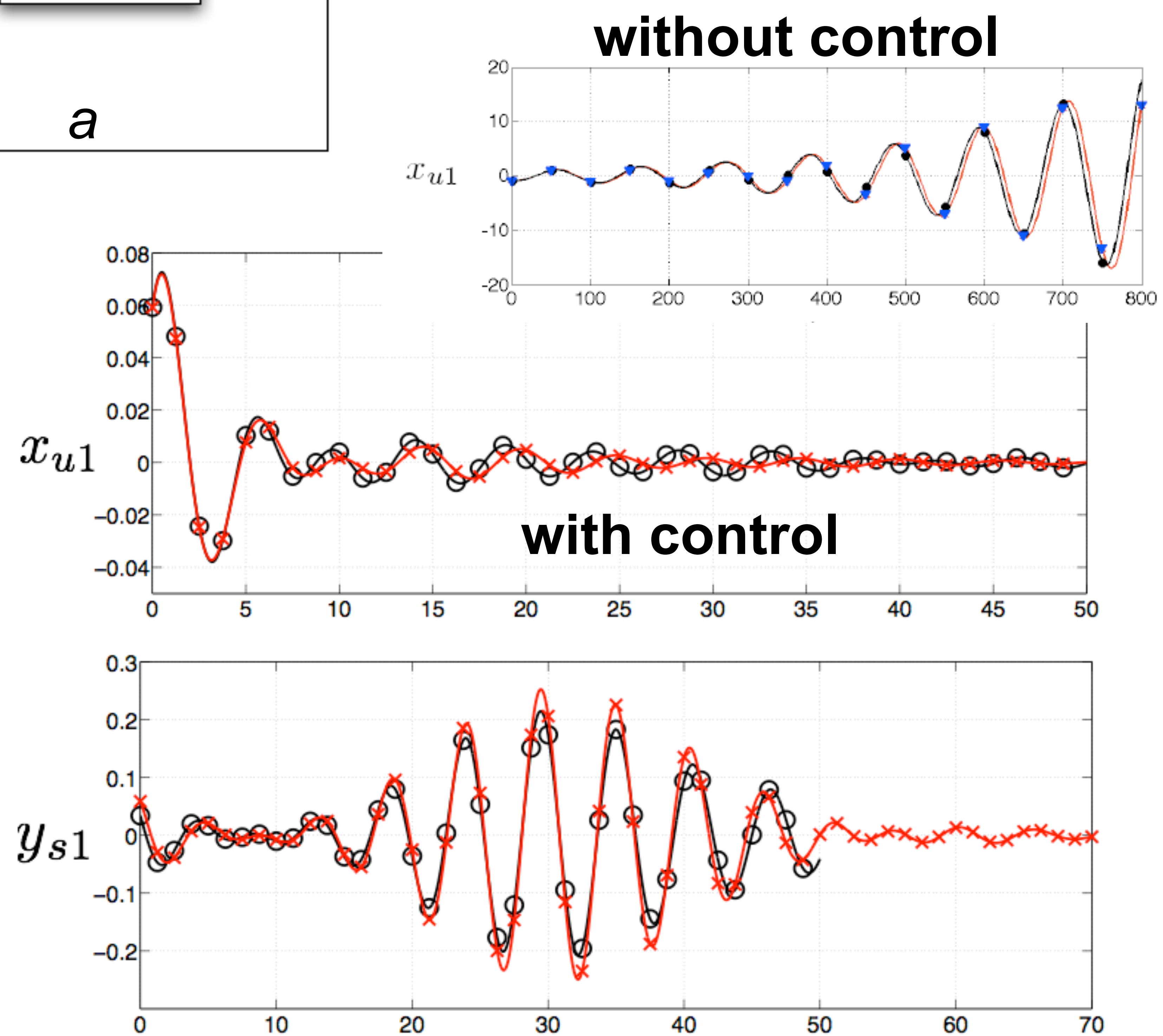
# Balancing modes: stable subspace

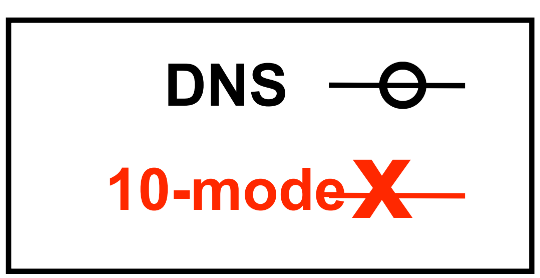


# Model results: controlled case



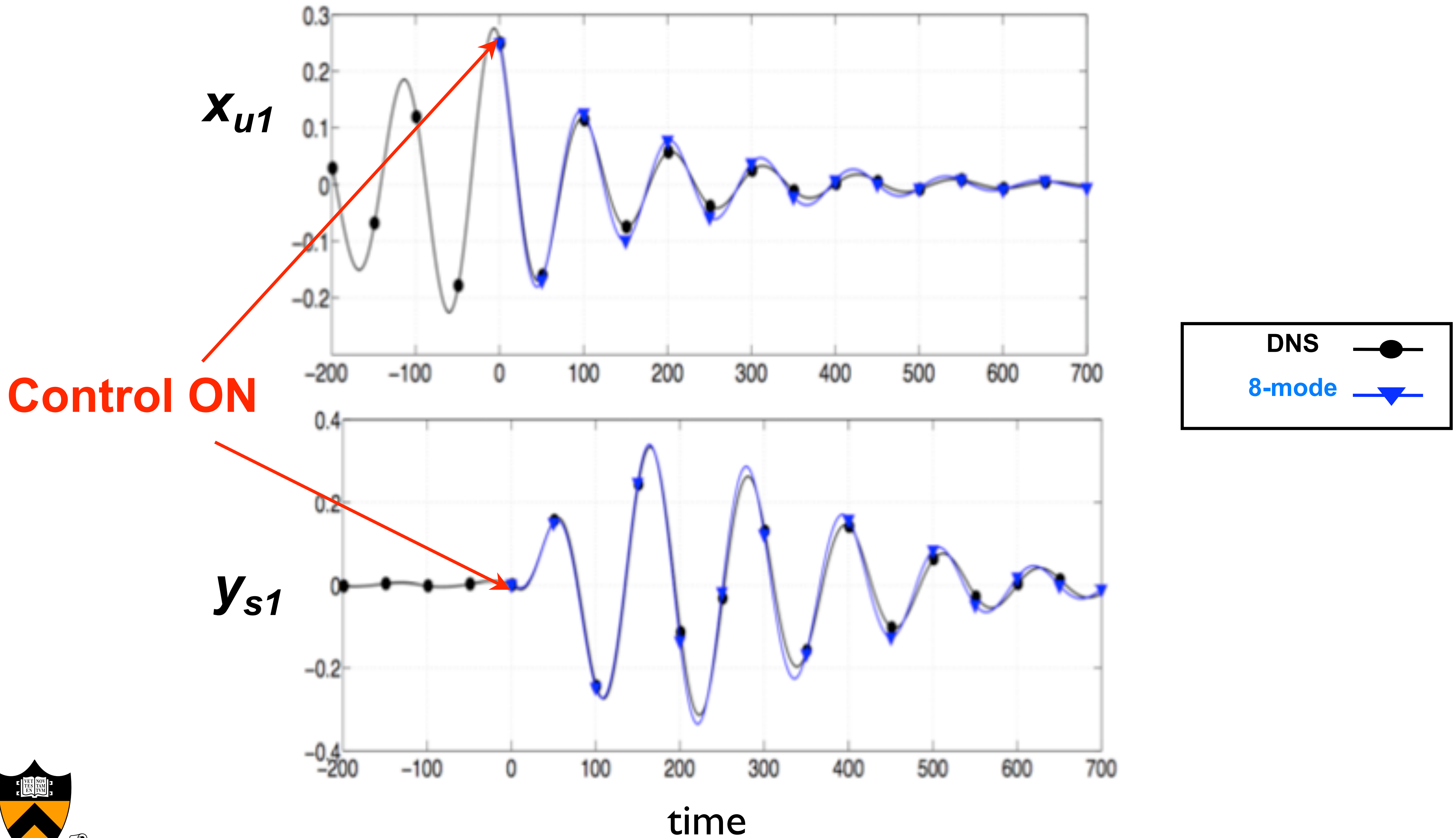

- Control based on a 10-mode model
- Gain K using LQR



# Control in full nonlinear system: close to steady state

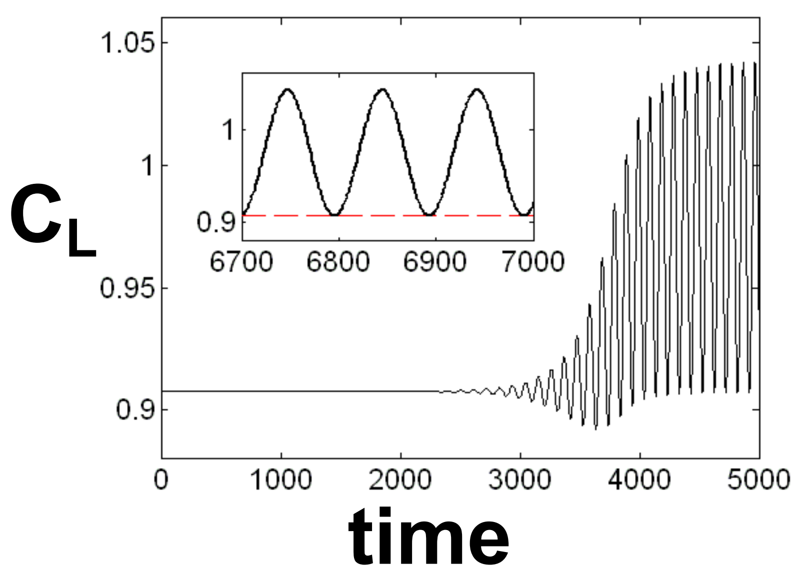
Results of an 8-mode model

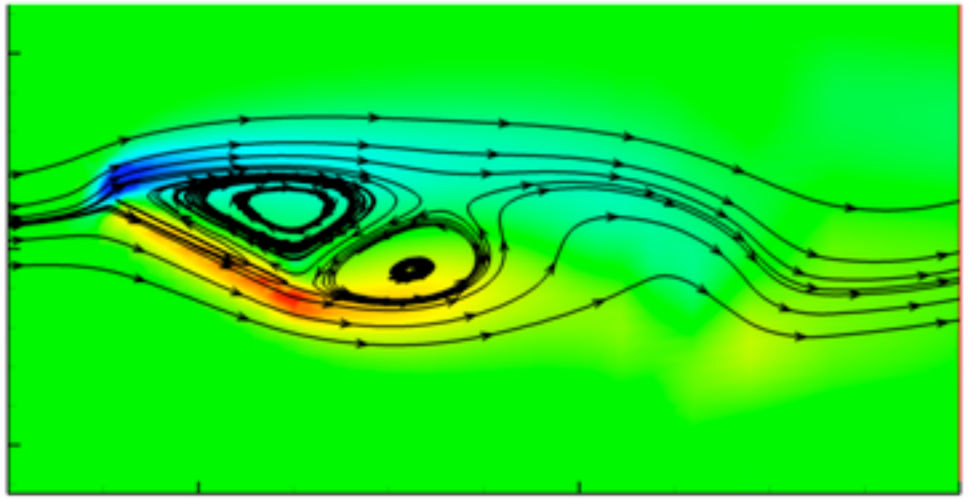


# Feedback stabilization at AoA=25

- Full state feedback
- Large domain of attraction even in the full NL system
- Controller suppresses the vortex shedding

No control



# Outline

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  - Importance of inner product for Galerkin projection
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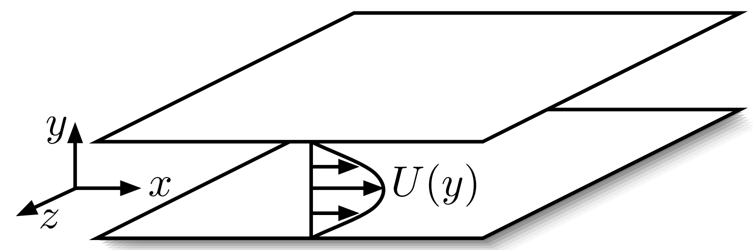
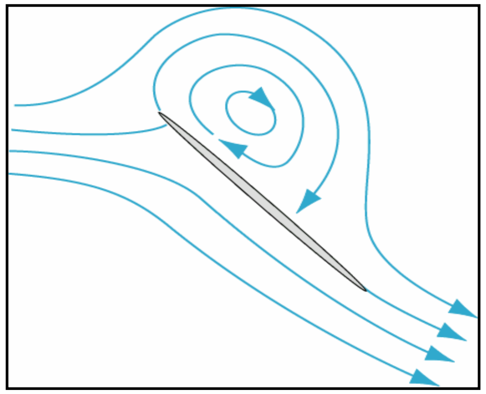
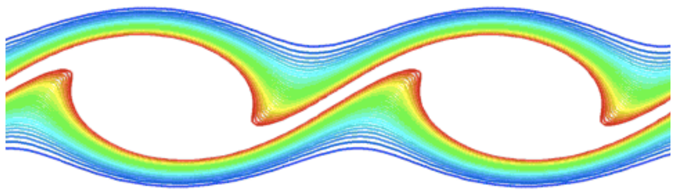
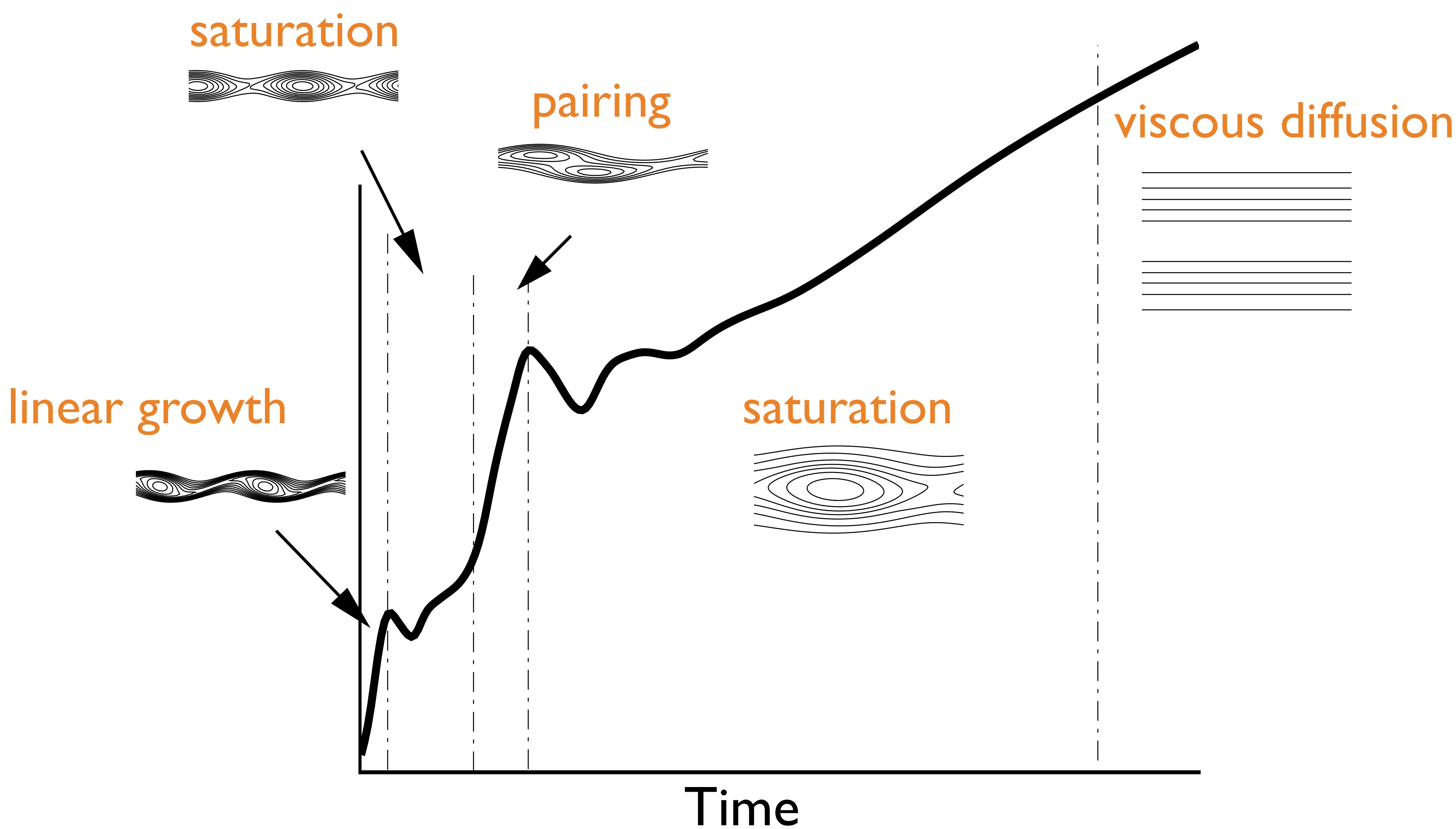




# Modeling free shear layers

- Evolution history of thickness for temporal shear layer (spatially periodic):




- Model initial linear growth, saturation, pairing, and eventual viscous diffusion

[Mingjun Wei, CW Rowley, JFM, to appear, 2008]

# Methodology

- Scale POD modes dynamically in y direction to account for shear layer spreading
- Scaling invariants:
  - divergence of velocity field
  - inner product
- Key idea: template fitting
- Main result: an equation for the shear layer spreading rate:
  - as usual, also get equations for time coefficients of POD modes

# Scaling basis functions

- Write solution in scaled reference frame

$$\mathbf{q} = (u, v)$$

$$\mathbf{q}(x, y, t) = G(g)\tilde{\mathbf{q}}(x, g(t)y, t)$$

- Choose $G(g) = \begin{bmatrix} 1 & 0 \\ 0 & 1/g \end{bmatrix}$: $\boxed{\text{div}\ \mathbf{q} = \text{div}\ \tilde{\mathbf{q}}}$
- Expand scaled variable $\tilde{\mathrm{q}}$ in terms of POD modes

$$\tilde{\mathbf{q}}(x, y, t) = \mathbf{u}_0(y) + \sum_{j=1}^{n} a_j(t)\varphi_j(x, y)$$

- Advantage of the scaling: capture similar-looking structures as shear layer spreads
- Advantage of divergence-invariant mapping: automatically satisfy continuity equation; simplify pressure term

# Template fitting

- How do we choose the scaling g(t)?
  - Choose g(t) so that $\tilde{\mathbf{q}}(x,y,t)$ lines up best with a preselected template (here, the base flow):

$$\frac{d}{ds}\bigg|_{s=0} \|\tilde{\mathbf{q}}(x,y,t) - \mathbf{u}_0(x,h(s)y)\|^2 = 0$$

for any curve $h(s) > 0$ with $h(0) = 1$

  - This means the scaled solution $\tilde{\mathbf{q}}(x,y,t)$ satisfies

$$\left\langle y\frac{\partial \mathbf{u}_0}{\partial y}, \tilde{\mathbf{q}} - \mathbf{u}_0 \right\rangle = 0$$

  - Geometrically, the set of all "properly scaled" functions $\tilde{\mathbf{q}}$ is an affine space through $\mathbf{u}_0$ and orthogonal to $y\partial_y \mathbf{u}_0$
  - This enables one to write dynamics for how the thickness g(t) evolves

$$\frac{\dot{g}}{g} = \frac{\langle f_g^1(\tilde{u}), y\partial_y u_0\rangle}{\langle y\partial_y \tilde{u}, y\partial_y u_0\rangle}$$

# Equation for evolution of the thickness

- How does g(t) evolve in time?
  - We have a constraint ($\tilde{\mathbf{q}}(x, y, t)$ lines up best with template $\mathbf{u}_0$):

$$\left\langle y\frac{\partial \mathbf{u}_0}{\partial y}, \tilde{\mathbf{q}} - \mathbf{u}_0 \right\rangle = 0$$

  - Differentiate:

$$\left\langle y\frac{\partial \mathbf{u}_0}{\partial y}, \frac{\partial \tilde{\mathbf{q}}}{\partial t} \right\rangle = 0$$

  - Use equations of motion

$$\frac{\partial \tilde{\mathbf{q}}}{\partial t} = f_g(\tilde{\mathbf{q}}) - \frac{\dot{g}}{g} y \frac{\partial \tilde{\mathbf{q}}}{\partial y} - G(1/g)\dot{G}(g, \dot{g})\tilde{\mathbf{q}}(x, y, t)$$

  - This gives an equation for g:

$$\boxed{\frac{\dot{g}}{g} = \frac{\langle f_g^1(\tilde{u}), y\partial_y u_0 \rangle}{\langle y\partial_y \tilde{u}, y\partial_y u_0 \rangle}}$$

# Results

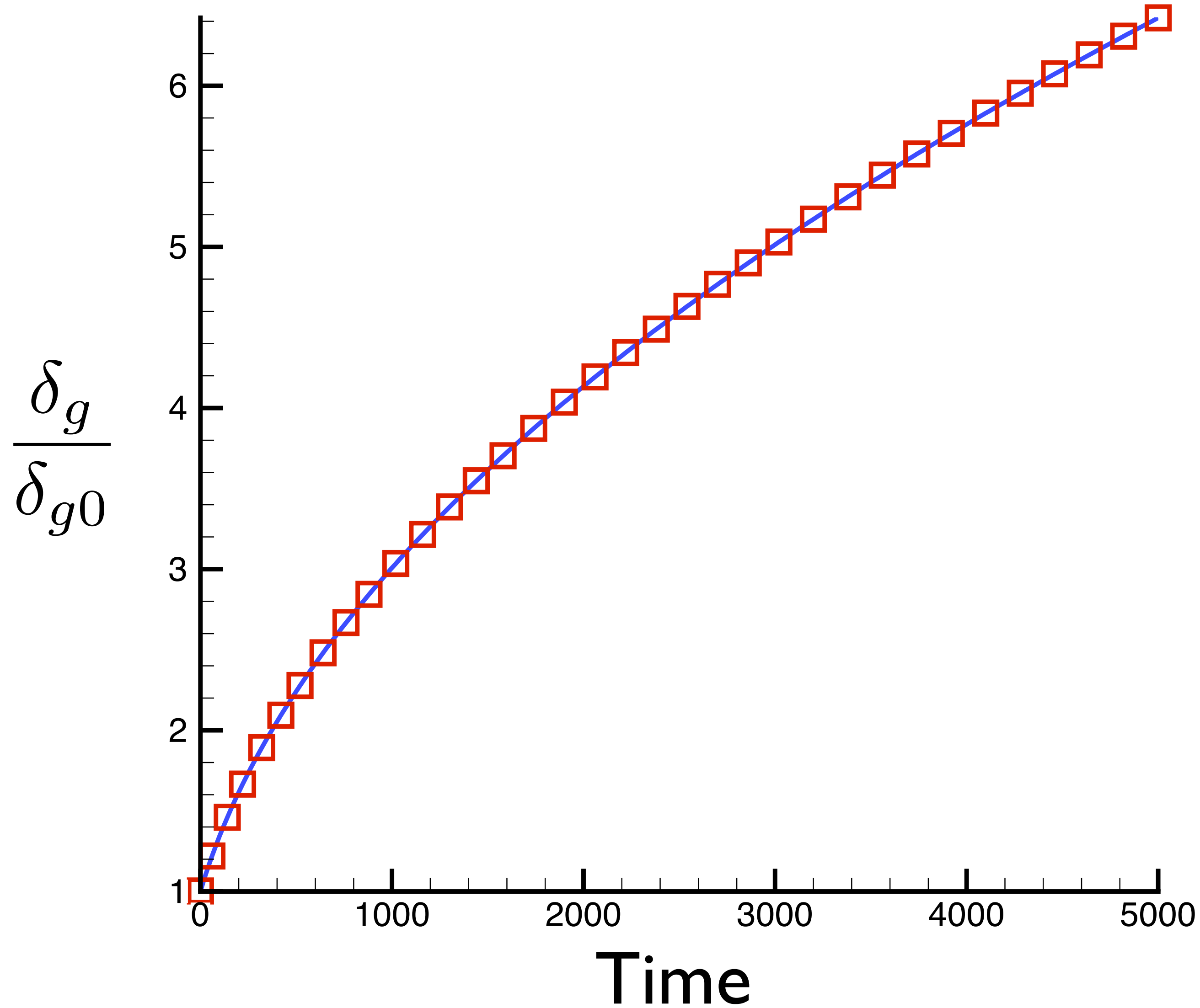
- Base flow with small perturbation
  - Base flow: $u_0 = U_c \mathrm{erfc}(\eta), \quad \eta = \frac{-y}{2g}\sqrt{\frac{\mathrm{Re}}{t_0}}$
  - Perturbation is along the unstable eigenfunction of the linearized problem
- Consider three separate cases
  - Self-similar solution (no perturbation)
  - [ Vortex roll-up transient (perturbation with k=1) ]
  - Vortex pairing transient (perturbation with k=2):
    - vortex roll-up
    - pairing
    - k=1 mode arises through pairing

# Model results: k=0

- Only one equation left for g:

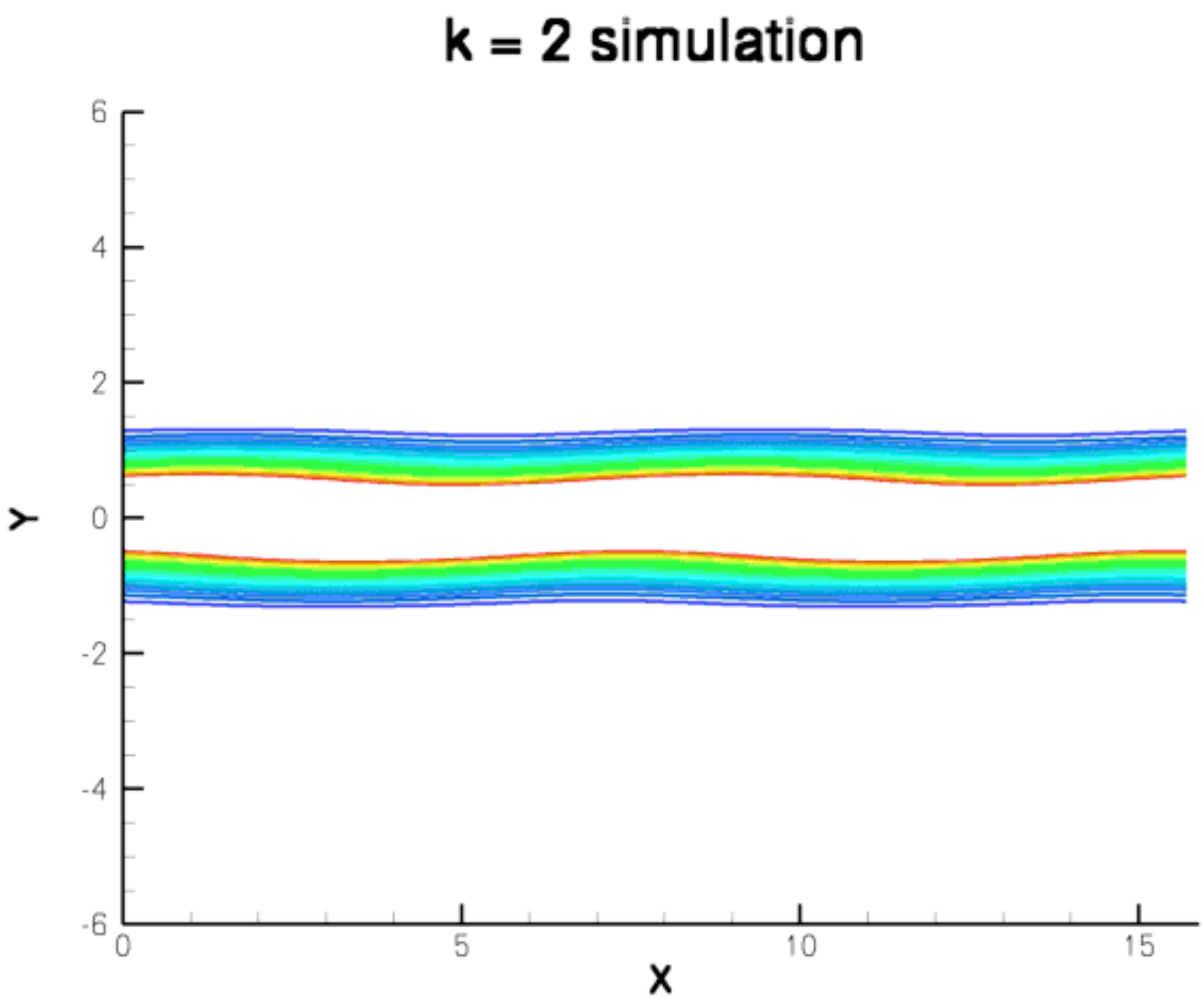
$$\dot{g} = \frac{1}{\mathrm{Re}}\frac{d_0}{n_0}g^3 \quad \Longrightarrow \quad \dot{g} = -\frac{g^3}{2t_0} \quad \Longrightarrow \quad g(t) = \sqrt{\frac{t_0}{t}}$$

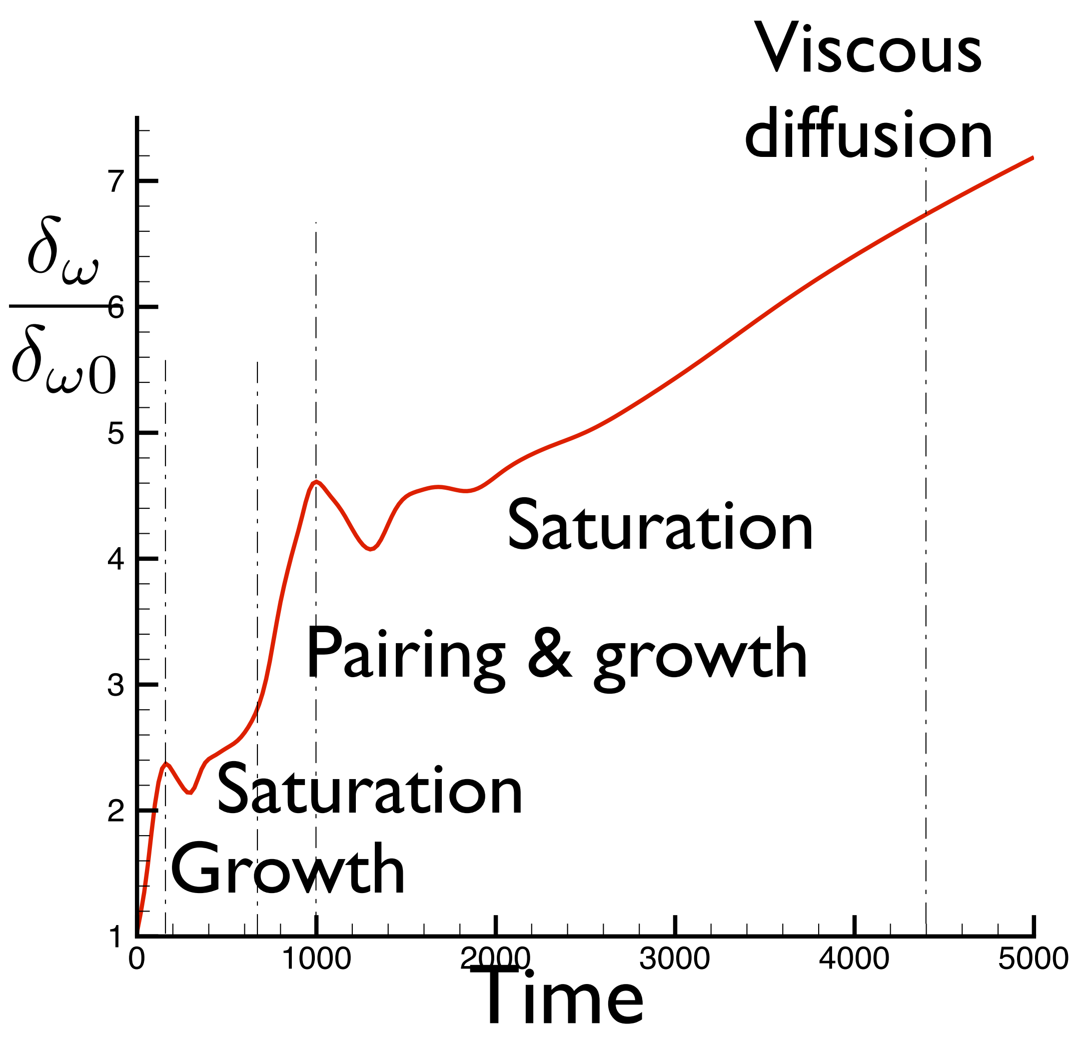
- Recovers exact theoretical growth rate for Stokes problem:



# Movie of DNS

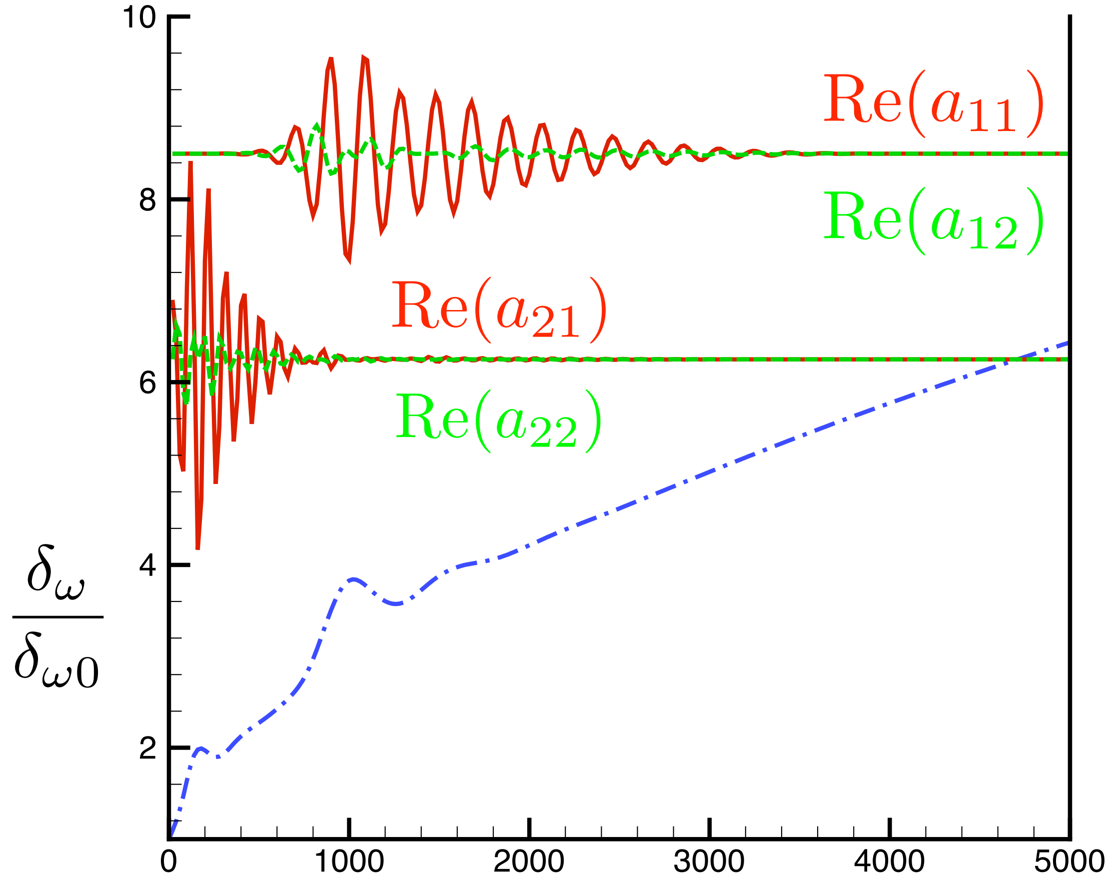
- Vortex pairing (initial condition with k=2)    Re = 200



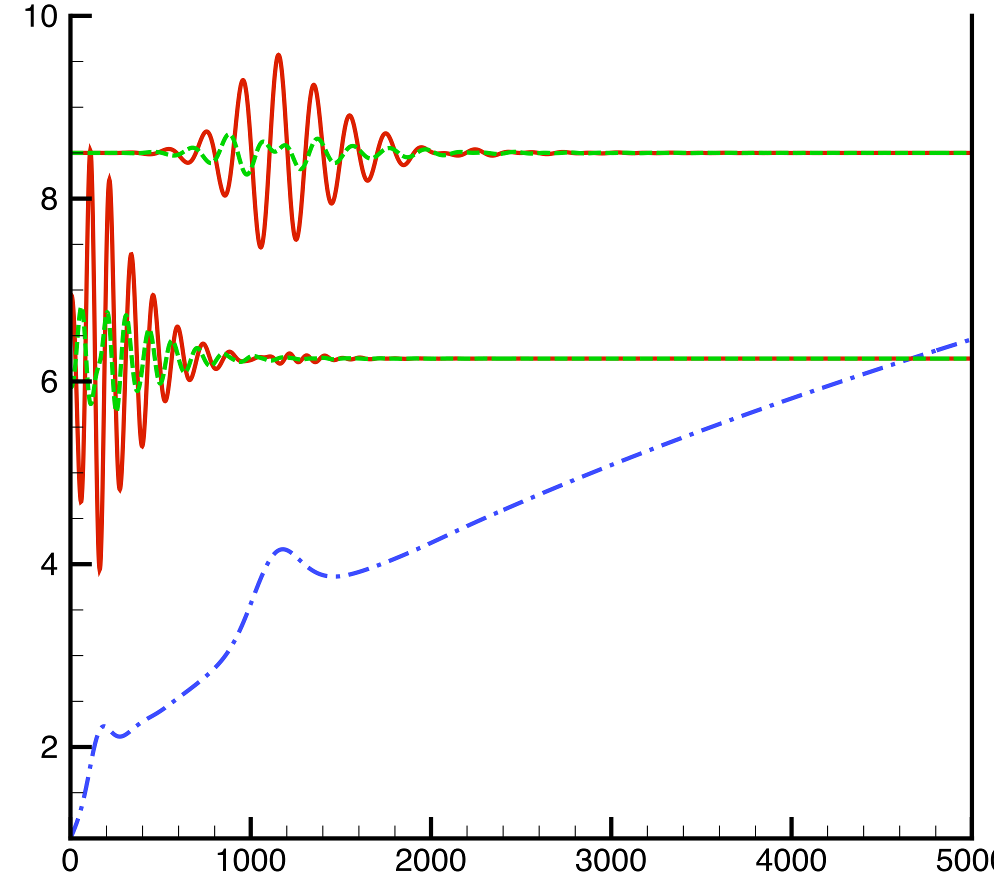



# Model results: k=2

- Thickness and amplitude of POD modes for k=2 initial condition: projection of full simulation

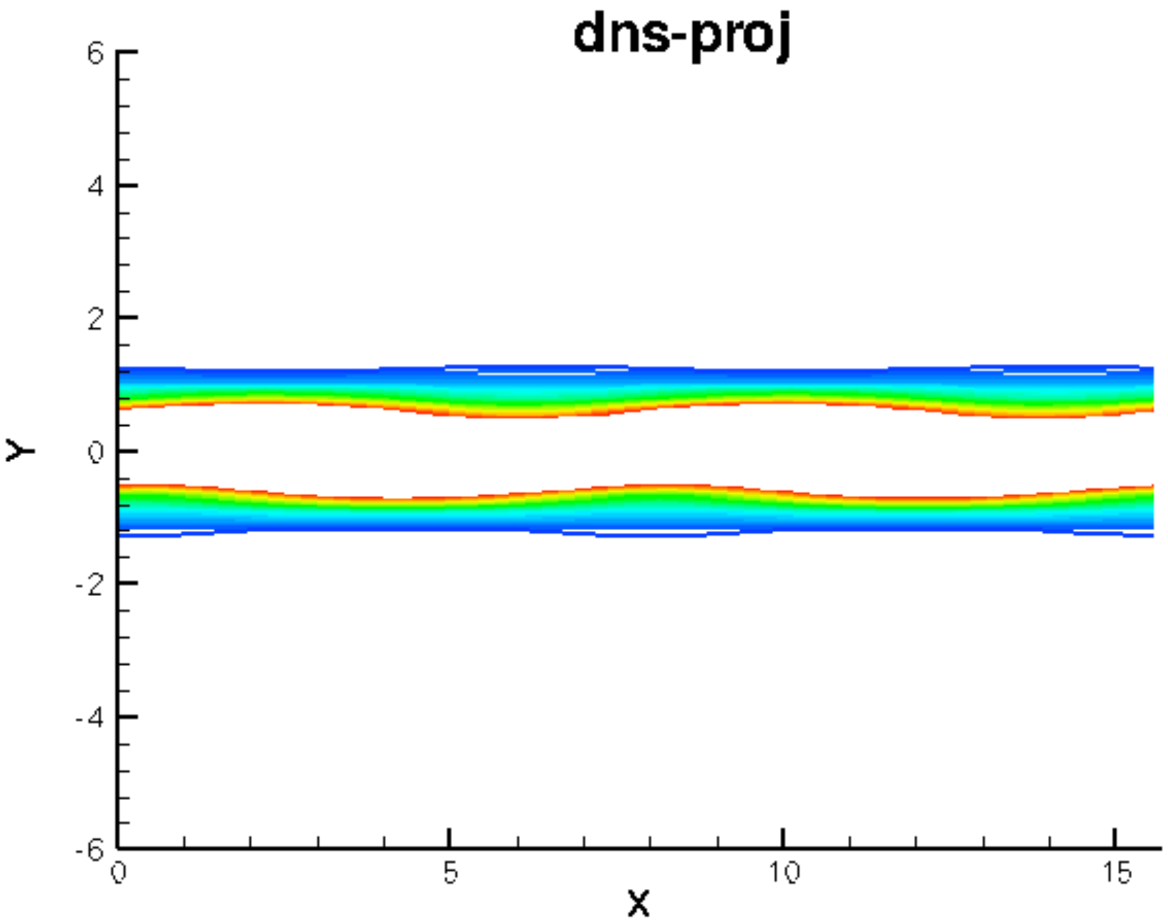


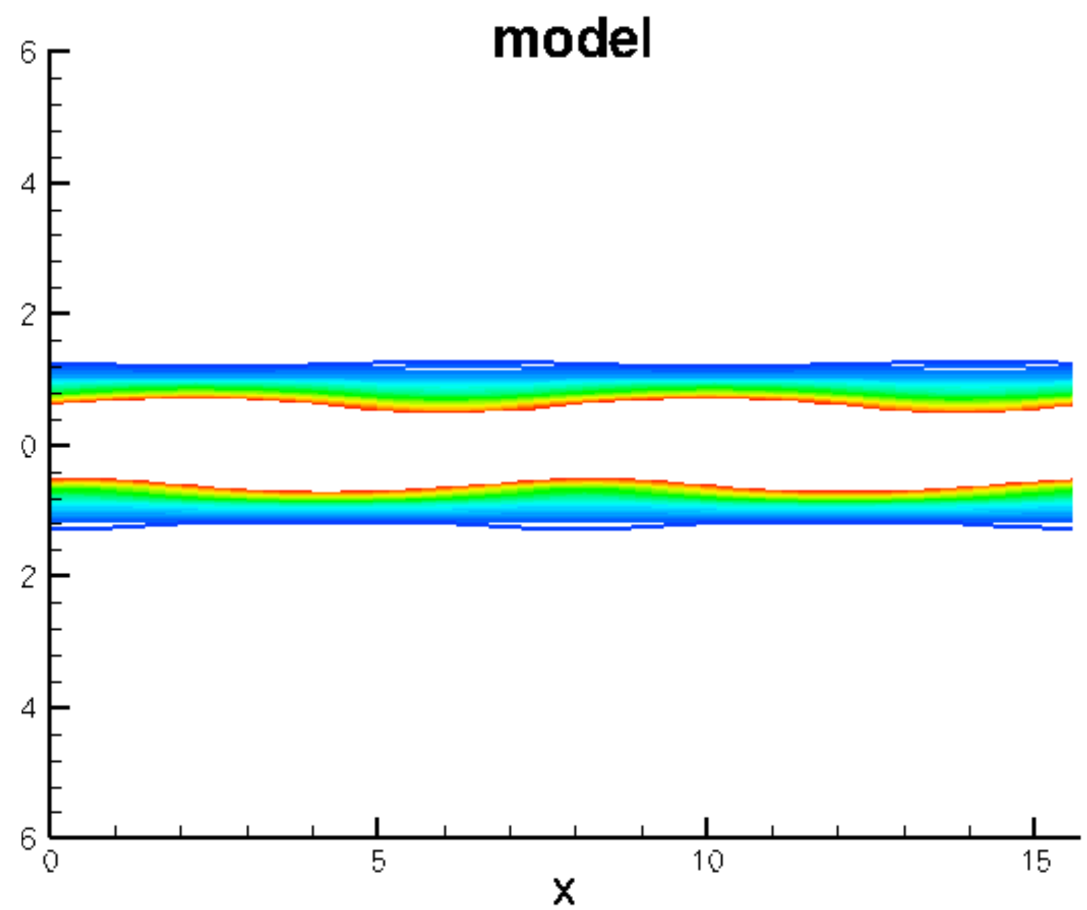

- Thickness and amplitude of POD modes for k=2 initial condition: low-dimensional model



# DNS v.s. Model

- Comparison of simulation and model results



# Summary

- Approximate balanced truncation
  - Approximates exact balanced truncation to as high accuracy as desired, using snapshots from linearized and adjoint simulations
  - Computational cost similar to POD, once snapshots computed
  - For a given number of modes, transients and frequency response much more accurately captured than POD models of same order
  - Extension of basic approach to model unstable linear systems
  - Feedback controllers designed from these models perform well, even on full-order, nonlinear systems
  - Extensions to (weakly) nonlinear systems straightforward
- Dynamically scaled POD modes
  - For systems with self-similar behavior, dynamic scaling decreases number of modes required
  - Temporal shear layer dynamics modeled with 4 complex modes, including linear growth, saturation, pairing, and viscous diffusion