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The sliding Couette flow problem

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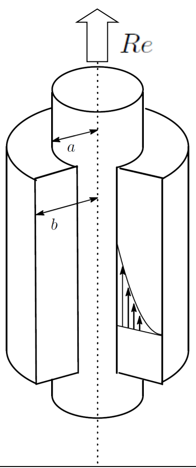
Background

Sliding Couette flow



- Linearly stable
  for η> 0.1415:Gittler(1993)
  (η : radius ratio)
- Nonlinear solution:
  not yet known
- Applications:
  ・catheter through blood vessel
  ・train through a tunnel

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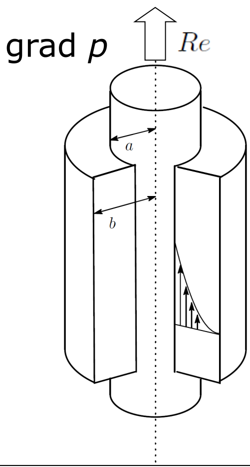


- plane Couette flow
  - $\eta \to 1$
- pipe flow
  - add axial pressure gradient
  - adjust Re
  - $\eta \to 0$
- moderate η

pCf with curvature + pf with solid core

Energy (stability) surface

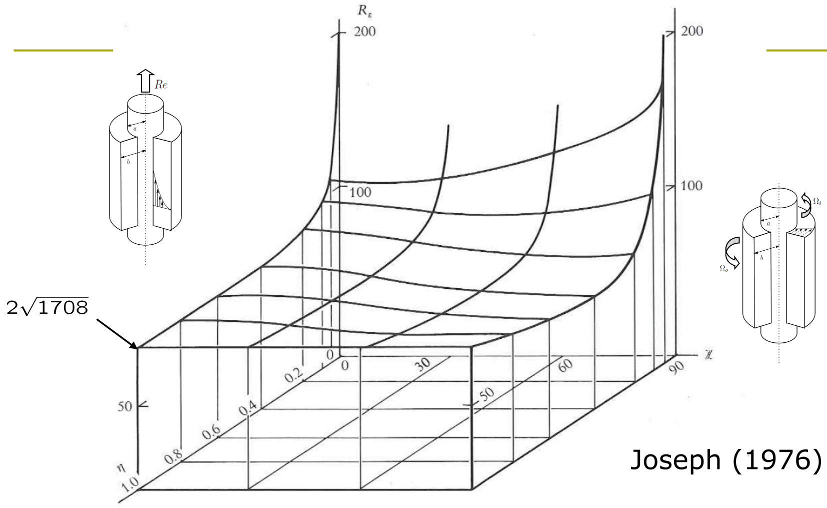
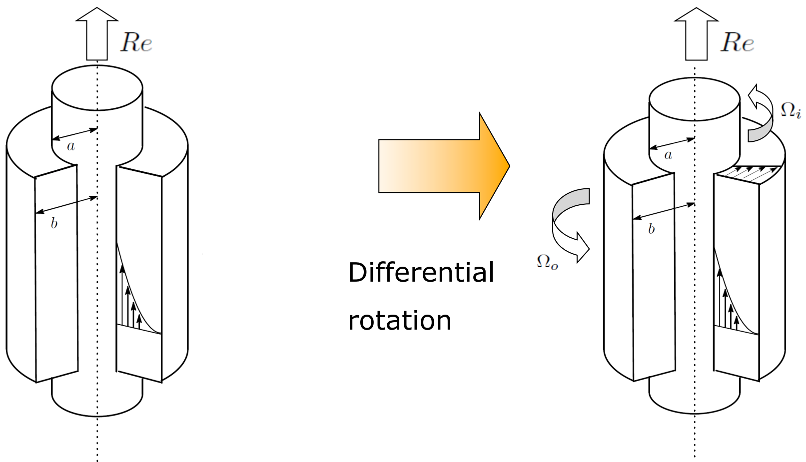


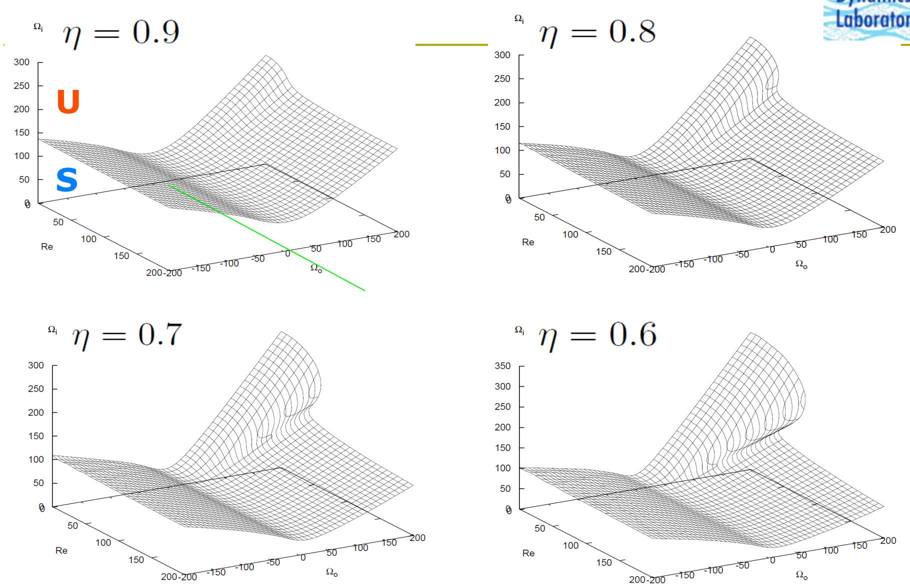
Fig. 52.1: Energy (stability) surface for Couette flow between rotating-sliding cylinders as a function of radius ratio, η, and the angle χ. Plane Couette flow in a rotating co-ordinate system is given by $\eta = 1$; Couette flow in an annulus with no differential rotation is given by $\chi = 0°$; and Taylor flow is given by $\chi = 90°$. Because the circumferential wave-number, n, must be an integer, the above smooth surface is an approximation to the surface with discontinuous first derivatives. The smoothed-out version is barely distinguishable from the true surface (Joseph and Munson, 1970)

Background

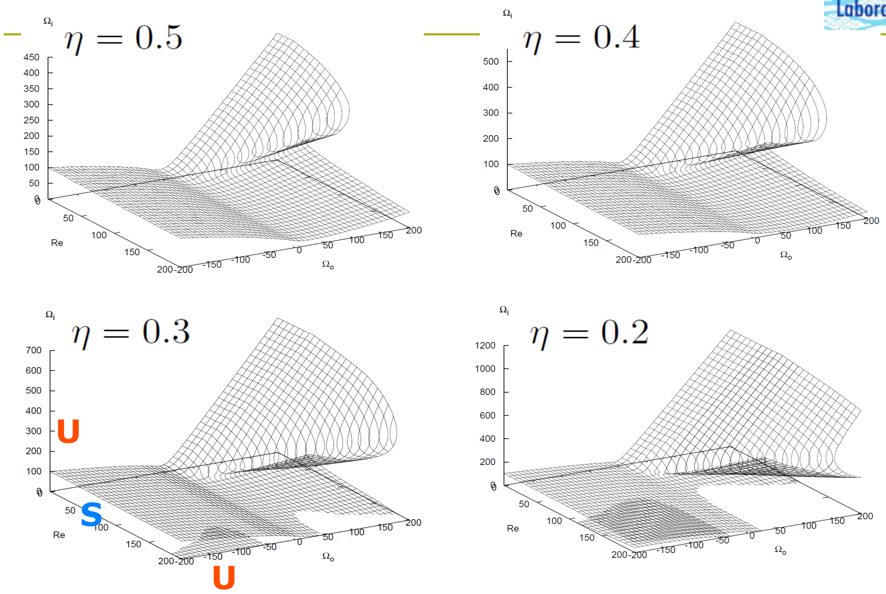
Sliding Couette flow **Spiral Couette Flow**



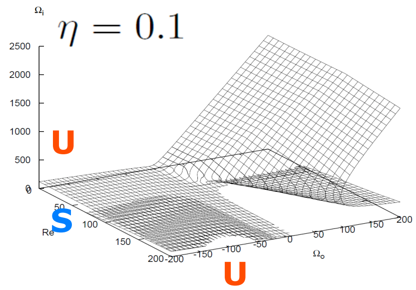
Neutral surface（1）



Neutral surface（2）



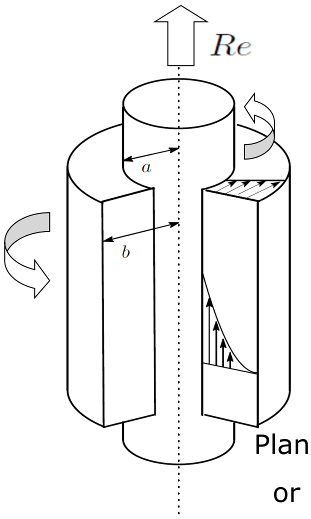
Neutral surface（3）



Re_L = 3.61E+06 (Ω_o= Ω_i = 0)

Deguchi & MN : in preparation

Spiral Couette flow



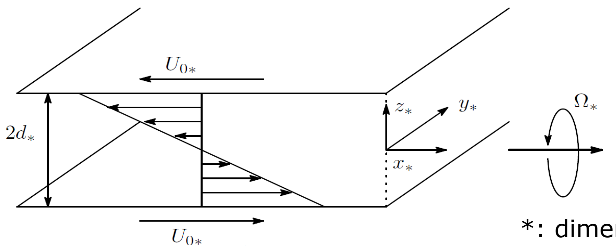
- Narrow gap limit
- Rigid-body rotation

Plane Couette fow with streamwise rotation

or Rotating plane Couette flow by Joseph (1976)

Model

- Incompressible fluid between two parallel plates with infinite extent
- Translational motion of the plates with opposite directions
- Constant streamwise system rotation



*: dimensional quantities

Energy method: Re_E

The *Reynolds-Orr* energy equation

$$\frac{1}{2}\frac{d}{dt_*}\left\langle |\boldsymbol{u}_*|^2 \right\rangle = -\nu_* \left\langle |\nabla_* \boldsymbol{u}_*|^2 \right\rangle - \langle \boldsymbol{u}_* \cdot (\boldsymbol{u}_* \cdot \nabla_*)\boldsymbol{u_{b*}} \rangle$$

Nondimensionalise:

$$\frac{1}{2}\frac{d}{dt}\left\langle |\boldsymbol{u}|^2 \right\rangle = -\left\langle |\nabla \boldsymbol{u}|^2 \right\rangle - \langle \boldsymbol{u} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u_b} \rangle$$

$$= -\left(\frac{\left\langle |\nabla \boldsymbol{u}|^2 \right\rangle}{\langle -\boldsymbol{u} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u_b} \rangle} - 1 \right) \langle -\boldsymbol{u} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u_b} \rangle$$

$$Re_E = \inf \left\{ \frac{\left\langle |\nabla \boldsymbol{u}|^2 \right\rangle}{\langle u_x u_z \rangle} \right\}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Energy method: Re_E

$$R(\boldsymbol{u}, \pi) = \frac{\langle |\nabla \boldsymbol{u}|^2 - 2\pi \nabla \boldsymbol{u} \rangle}{\langle u_x u_z \rangle}$$

Euler-Lagrange equation $\quad \nabla^2 \boldsymbol{u} - \nabla \pi + \frac{1}{2} R(u_z \hat{\boldsymbol{i}} + u_x \hat{\boldsymbol{k}}) = 0$

$(\frac{\partial}{\partial x} = 0, \phi \propto e^{i\beta y} f(z))$

eliminating ψ

$$\left\{ \left(\frac{\partial^2}{\partial z^2} - \beta^2 \right)^3 + \frac{1}{4} R^2 \beta^2 \right\} f(z) = 0$$

boundary condition

$$\phi = \frac{\partial \phi}{\partial z} = \left(\frac{\partial^2}{\partial z^2} - \beta^2 \right)^2 \phi = 0 \quad at \quad z = \pm 1$$

$Re_E = 20.6625$

β=1.558

Governing equations

time scale: d_*^2/ν_* length scale: d_* velocity scale: ν_*/d_*

- equation of continuity

$$\nabla \cdot \boldsymbol{u} = 0$$

- momentum equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla \Pi + \nabla^2 \boldsymbol{u} - \boldsymbol{\Omega} \times \boldsymbol{u}$$

- boundary condition

$$u = \mp R_x \quad at \quad z = \pm 1$$

- basic solution

$$u_b(z) = -R_x z$$

Reynolds number $R_x = \dfrac{U_{0*} d_*}{\nu_*}$

Rotation rate $\Omega = \dfrac{2\Omega_* d_*^2}{\nu_*}$

Disturbance equatuins

separation of the velocity field into basic flow, mean flows and residual

$$\boldsymbol{u} = \boldsymbol{u_b} + \breve{U}(z)\boldsymbol{i} + \breve{V}(z)\boldsymbol{j} + \breve{\boldsymbol{u}}$$

Further decomposition of the residual into poloidal and toroidal parts

$$\breve{\boldsymbol{u}} = \nabla \times \nabla \times (\phi \boldsymbol{k}) + \nabla \times (\psi \boldsymbol{k})$$

substitute into the momentum equation

$\boldsymbol{k} \cdot (\nabla \times$

$\boldsymbol{k} \cdot (\nabla \times \nabla \times$

$$\frac{\partial}{\partial t}\nabla^2 \Delta_2 \phi = \left(\nabla^4 + (u_b + \breve{U})'' \frac{\partial}{\partial x} + \breve{V}'' \frac{\partial}{\partial y} - (u_b + \breve{U}) \frac{\partial}{\partial x}\nabla^2 - \breve{V} \frac{\partial}{\partial y}\nabla^2\right) \Delta_2 \phi - \Omega_x \frac{\partial}{\partial x}\Delta_2 \psi - \boldsymbol{k} \cdot \nabla \times \nabla \times [\breve{\boldsymbol{u}} \cdot \nabla \breve{\boldsymbol{u}}]$$

$$\frac{\partial}{\partial t}\Delta_2 \psi = \left((u_b + \breve{U})' \frac{\partial}{\partial y} + \Omega_x \frac{\partial}{\partial x} - \breve{V}' \frac{\partial}{\partial x}\right) \Delta_2 \phi + \left(\nabla^2 - (u_b + \breve{U}) \frac{\partial}{\partial x} - \breve{V} \frac{\partial}{\partial y}\right) \Delta_2 \psi + \boldsymbol{k} \cdot \nabla \times [\breve{\boldsymbol{u}} \cdot \nabla \breve{\boldsymbol{u}}]$$

Mean flow equations

x, y-averages of the x, yomponents of the momentum equation

$$\check{U}'' + \frac{\partial}{\partial z}\overline{\Delta_2\phi\left(\frac{\partial^2}{\partial z\partial x}\phi + \frac{\partial}{\partial y}\psi\right)} = \frac{\partial \check{U}}{\partial t}$$

$$\check{V}'' + \frac{\partial}{\partial z}\overline{\Delta_2\phi\left(\frac{\partial^2}{\partial z\partial y}\phi - \frac{\partial}{\partial x}\psi\right)} = \frac{\partial \check{V}}{\partial t}$$

$$\overline{*} = \frac{\alpha\beta}{4\pi^2}\int_0^{2\pi/\alpha}\int_0^{2\pi/\beta} * dxdy$$

Boundary conditions:

$$\check{U} = \check{V} = \phi = \frac{\partial \phi}{\partial z} = \psi = 0 \quad at \quad z = \pm 1$$

Linear stability analysis

Perturbation equations

$$\frac{\partial}{\partial t}\nabla^2\Delta_2\phi = \left(\nabla^4 + u_b''\frac{\partial}{\partial x} - u_b\frac{\partial}{\partial x}\nabla^2\right)\Delta_2\phi - \Omega_x\frac{\partial}{\partial x}\Delta_2\psi$$

$$\frac{\partial}{\partial t}\Delta_2\psi = \left(u_b'\frac{\partial}{\partial y} + \Omega_x\frac{\partial}{\partial x}\right)\Delta_2\phi + \left(\nabla^2 - u_b\frac{\partial}{\partial x}\right)\Delta_2\psi$$

expansion of ϕ, ψ

$$\phi = \sum_{l=0}^{L} a_l(1-z^2)^2 T_l(z)\exp[i\alpha x + i\beta y + \sigma t]$$

$$\psi = \sum_{l=0}^{L} b_l(1-z^2) T_l(z)\exp[i\alpha x + i\beta y + \sigma t]$$

$T_l(z)$: Chebyshev polynomials

Numerical method (Linear analysis)

Evaluation of ϕ, ψ at the collocation points

$$z_i = \cos\left(\frac{i\pi}{L+2}\right) \qquad (i = 1, 2, \cdots, L+1)$$

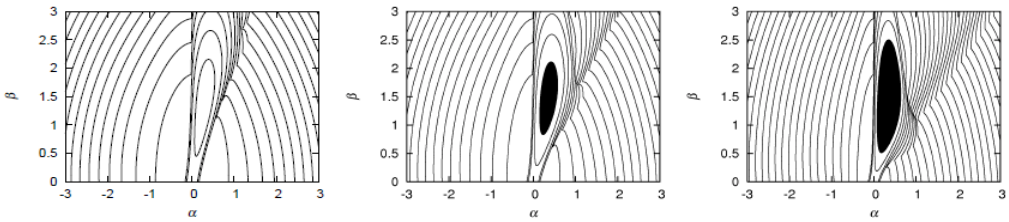
$$\boldsymbol{Ax} = \sigma \boldsymbol{Bx} \qquad \boldsymbol{x} = (\boldsymbol{a}, \boldsymbol{b})^T$$

Eigenvalue problem

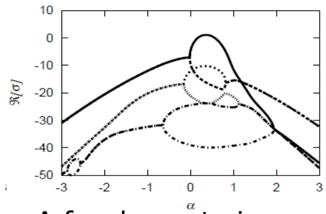
$\Re(\sigma) > 0$	unstable
$\Re(\sigma) = 0$	neutral
$\Re(\sigma) < 0$	stable

Results (Ω=40)

Unstable region: dark



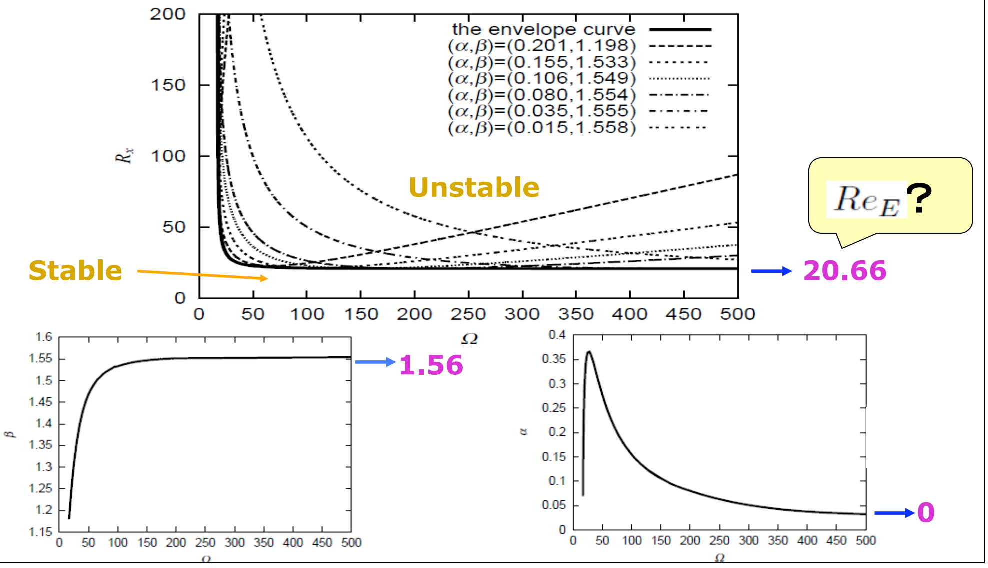
$R_x = 20$ $\qquad$ $R_x = 30$ $\qquad$ $R_x = 50$



A few largest eigenvalues

$R_x = 30 \qquad \beta = 1.5$

Results （neutral curves）



Results

The critical wave number and the critical Reynolds number

Ω	Critical wave number (α_c, β_c)	Critical Reynolds number
18	(0.201,1.198)	75.770525
20	(0.301,1.233)	45.802127
30	(0.362,1.358)	26.961652
40	(0.321,1.427)	23.798557
50	(0.278,1.468)	22.573006
100	(0.155,1.533)	21.112391
200	(0.080,1.554)	20.773511
500	(0.032,1.554)	20.680340

Long wave limit with constant β

consider $\Omega \longrightarrow \infty, \alpha \longrightarrow 0$

$$0 = \left(\nabla^4 + R_x z \frac{\partial}{\partial x} \nabla^2\right) \Delta_2 \phi - \Omega_x \frac{\partial}{\partial x} \Delta_2 \psi$$

$$0 = \left(-R_x \frac{\partial}{\partial y} + \Omega_x \frac{\partial}{\partial x}\right) \Delta_2 \phi + \left(\nabla^2 + R_x z \frac{\partial}{\partial x}\right) \Delta_2 \psi$$

$$\Omega \sim \Omega_{-1} \frac{1}{\alpha} + \Omega_0 + \Omega_1 \alpha + \Omega_2 \alpha^2 \cdots$$

$$R_x \sim R_0 + R_1 \alpha + R_2 \alpha^2 \cdots$$

$$\phi \sim \phi_0 + \phi_1 \alpha + \phi_2 \alpha^2 \cdots$$

$$\psi \sim \psi_0 + \psi_1 \alpha + \psi_2 \alpha^2 \cdots$$

Long wave limit with constant β

α^0:

$$\nabla^4 \Delta_2 \phi_0 - i \Omega_{-1} \Delta_2 \psi_0 = 0$$

$$\nabla^2 \Delta_2 \psi_0 + i \left(\Omega_{-1} - R_0 \beta\right) \Delta_2 \phi_0 = 0$$

$$\left\{ \left(\frac{\partial^2}{\partial z^2} - \beta^2\right)^3 + \Omega' \left(R_0 - \Omega'\right) \beta^2 \right\} \phi_0(z) = 0$$

boundary condition

$$\phi = \frac{\partial \phi}{\partial z} = \left(\frac{\partial^2}{\partial z^2} - \beta^2\right)^2 \phi = 0 \quad at \quad z = \pm 1 \qquad \frac{\Omega_{-1}}{\beta} = \Omega'$$

Re_L=20.6625
$\Omega' = 10.3312$
β=1.558

$Re_E = 20.6625$

Non-linear Analysis

Expand:

$$\phi = \sum_{l=0}^{L} \sum_{\substack{m=-M \\ (m,n)\neq(0,0)}}^{M} \sum_{n=-N}^{N} a_{lmn}\ (1-z^2)^2 T_l(z) \exp(im\alpha x + in\beta y)$$

$$\psi = \sum_{l=0}^{L} \sum_{\substack{m=-M \\ (m,n)\neq(0,0)}}^{M} \sum_{n=-N}^{N} b_{lmn}\ (1-z)^2 T_l(z) \exp(im\alpha x + in\beta y)$$

$$\check{U}(z) = \sum_{l=0}^{L} c_l\ (1-z)^2 T_l(z)$$

$$\check{V}(z) = \sum_{l=0}^{L} d_l\ (1-z)^2 T_l(z)$$

Boundary condition

$$\check{U} = \check{V} = \phi = \frac{\partial \phi}{\partial z} = \psi = 0 \quad at \quad z = \pm 1$$

Non-linear Analysis

Evaluate at

$$z_i = \cos\left(\frac{i\pi}{L+2}\right) \quad , \quad (i = 1, 2, \cdots, L+1)$$

$$A_{ij}x_j + B_{ijk}x_j x_k = 0, x_j \in (a_{lmn}, b_{lmn}, c_l, d_l)$$

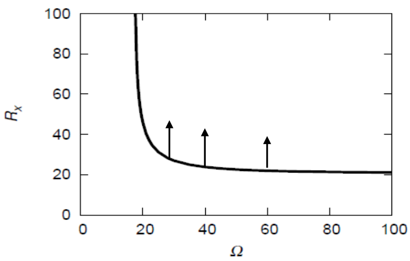
Solve by Newton-Raphson method

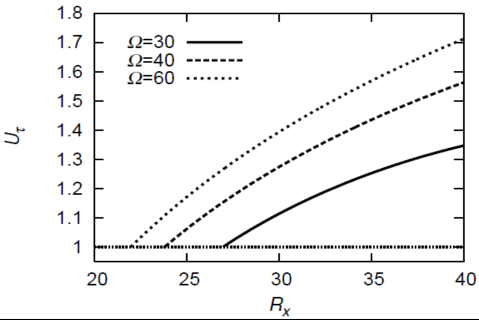
Nonlinear solution (bifurcation diagram)

momentum transport

$$U_\tau = \left.\frac{\partial \breve{U}}{\partial z}\right|_{z=1}$$

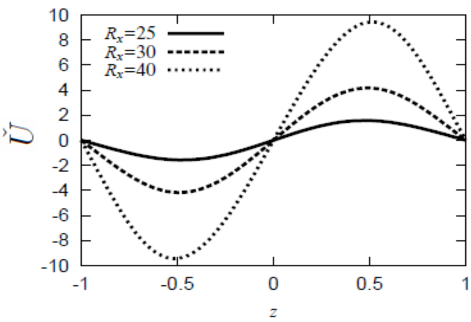
Ω	α	β	Critical Reynolds number
30	0.362	1.358	26.9616
40	0.321	1.427	23.7985
60	0.242	1.492	21.9562



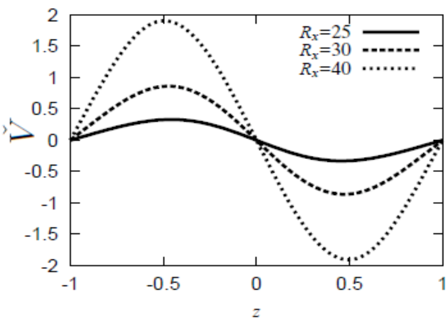


Mean flows

$$(\alpha, \beta) = (0.242, 1.492), \Omega = 60$$



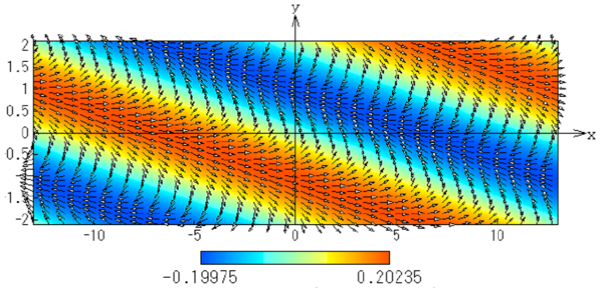
The mean flow $\breve{U}$



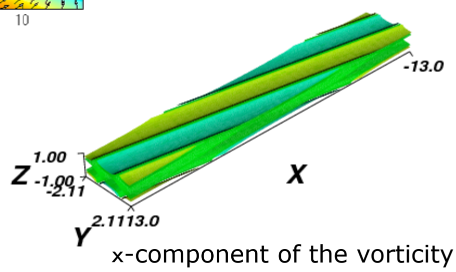
The mean flow $\breve{V}$

Steady spiral solution

$(\alpha, \beta) = (0.242, 1.492), R = 22, \Omega = 40$



flow field $(z = 0.5)$



x-component of the vorticity

Summary

Stability analysis:

- Basic state is stable for small rotation and becomes unstable at Ω ~17 to 3D perturbations.
- As Ω is increased, Re_Lcreases.
- $Re_E = Re_L$n the limit of Ω→∞ and α→0.

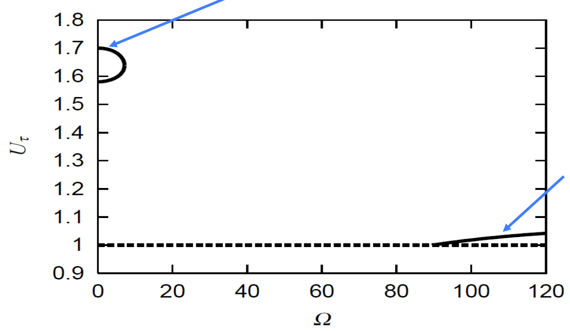
Nonlinear analysis

- Steady spiral solution bifurcates supercritically as a secondary flow.
- The mean flow in the spanwise direction is created.

Future work

- Connection with the 3-D solution of plane Couette flow without rotation

$(\alpha, \beta) = (0.5, 1.5)$
$R_x = 165$



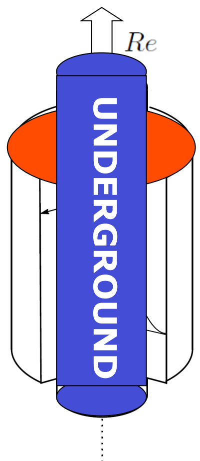
steady spiral solution

- Connection with cylinderical geometry

End

Background

Sliding Couette flow



- Linearly stable

  for η> 0.14.15:Gittler(1993)

  (η : radius ratio)

- Nonlinear solution:

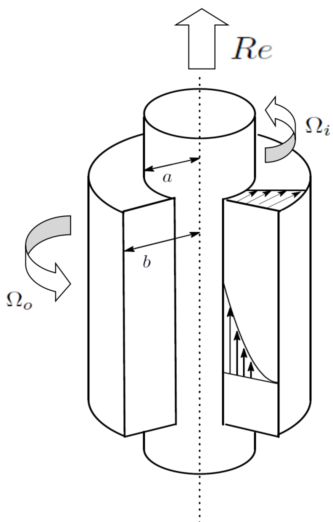
  not yet known

- Applications:

  ･catheter through blood vessel

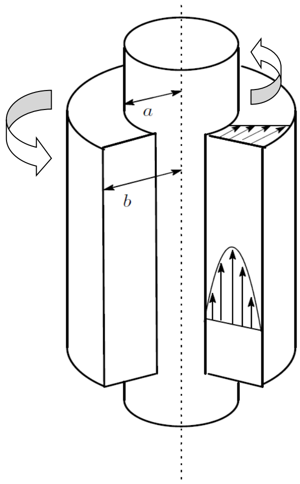
  ･train through a tunnel

Spiral Couette flow



- radius ratio: η

Spiral Poiseuille flow



Spiral Poiseuille flow (Joseph)

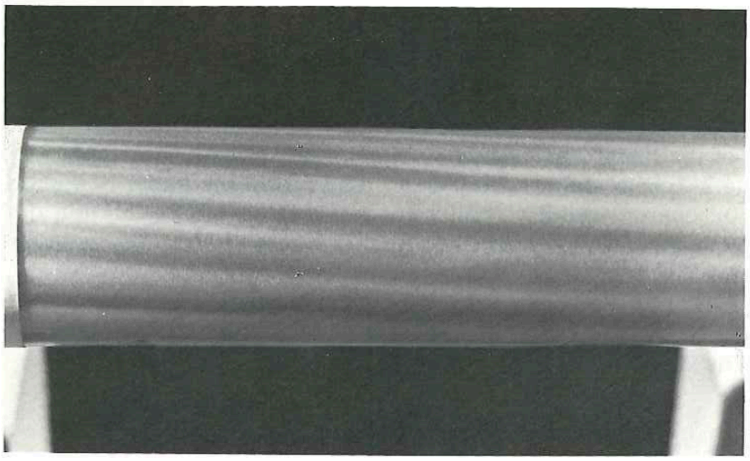


Fig. 46.3: Spiral vortices in rotating Poiseuille flow down the annulus with $\eta = 0.625$. The flow is from right to left. The operating conditions for the flow are $N_{R\theta} = 3120$, $N_{RZ} = 220$ and $\Gamma = 14.18$ (Nagib, 1972)

Spiral Poiseuille flow (Joseph)

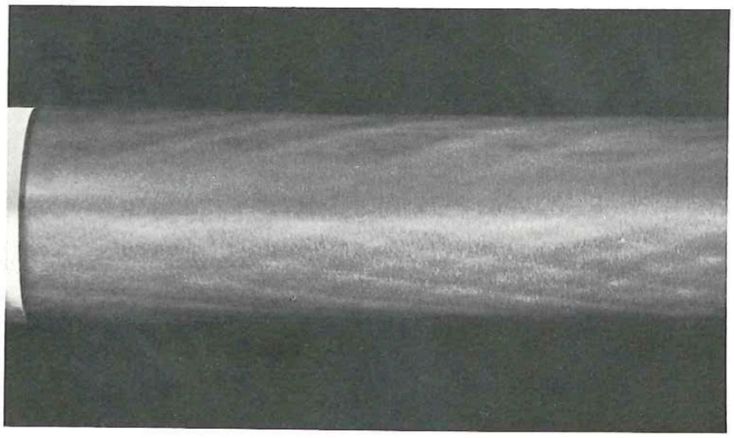
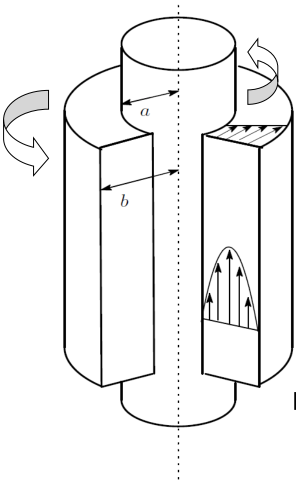


Fig. 46.4: Spiral vortices in rotating Poiseuille flow down the annulus. This figure shows two sets of vortices: an outer set and an inner set. The outer set is near to the outer cylinder and it dominates this photograph but less strongly than in Fig. 46.3. The spiral vortices near the inner wall are also visible. The two sets appear to make an equal and opposite angle with the axis. The flow is from right to left. The operating conditions for the flow are $N_{R\theta}=2000$, $N_{RZ}=340$ and $\Gamma=5.88$ (Nagib, 1972)

Spiral Poiseuille flow



- Narrow gap limit
- Rigid-body rotation

Plane Poiseuille fow with streamwise rotation

Masuda, Fukuda & Nagata (2008)

Governing equations

- equation of continuity

$$\nabla_* \cdot \boldsymbol{u}_* = 0$$

- momentum equation

$$\frac{\partial \boldsymbol{u}_*}{\partial t_*} + (\boldsymbol{u}_* \cdot \nabla_*)\boldsymbol{u}_* = -\frac{1}{\rho_*}\nabla_* \Pi_* + \nu_* \nabla_*^2 \boldsymbol{u}_* - 2\boldsymbol{\Omega}_* \times \boldsymbol{u}_*$$

- boundary condition

$$u_* = \mp U_{0*} \quad at \quad z_* = \pm 1$$

Coriolis force

$$\boldsymbol{\Omega}_* = (\Omega_{x*}, 0, 0)$$

ρ_* density

ν_* kinematic viscosity

Π_* pressure