FLUBIO
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An adjoint-based analysis of the secondary instability of the wake of a circular cylinder

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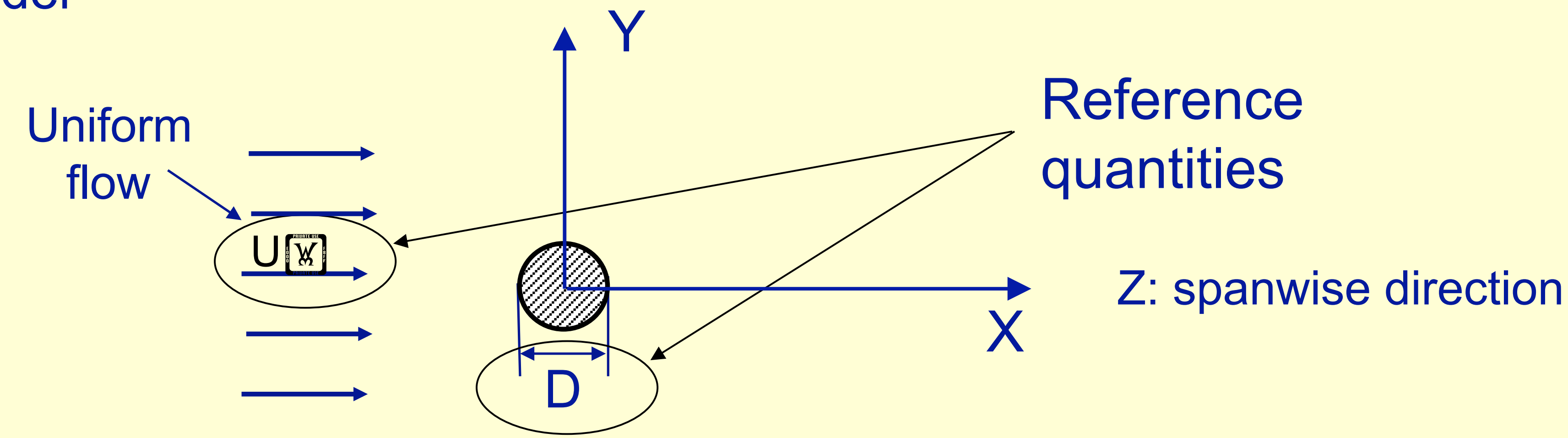
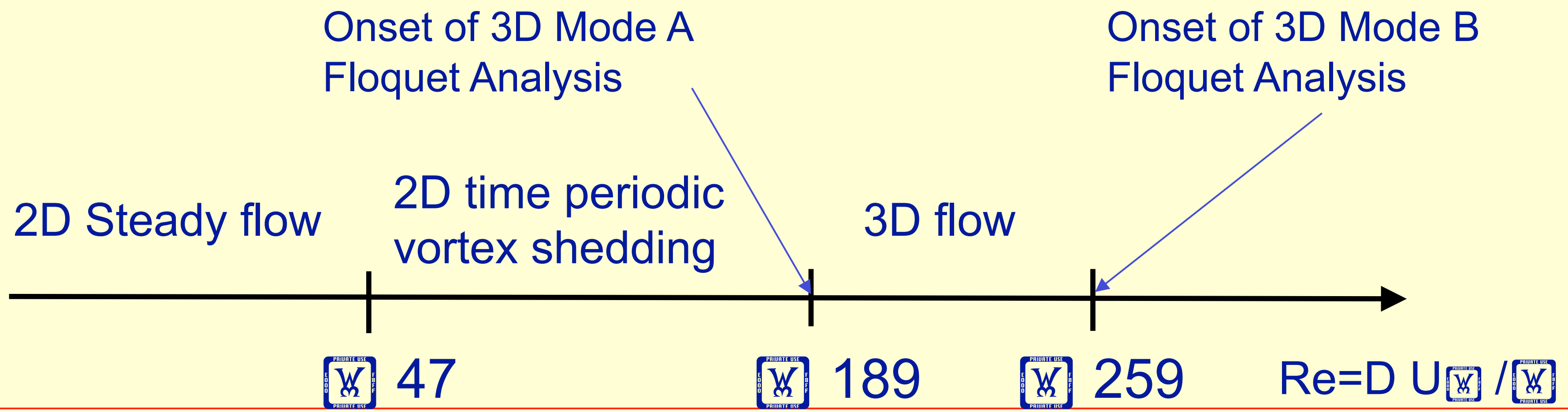
Flow configuration and main flow features

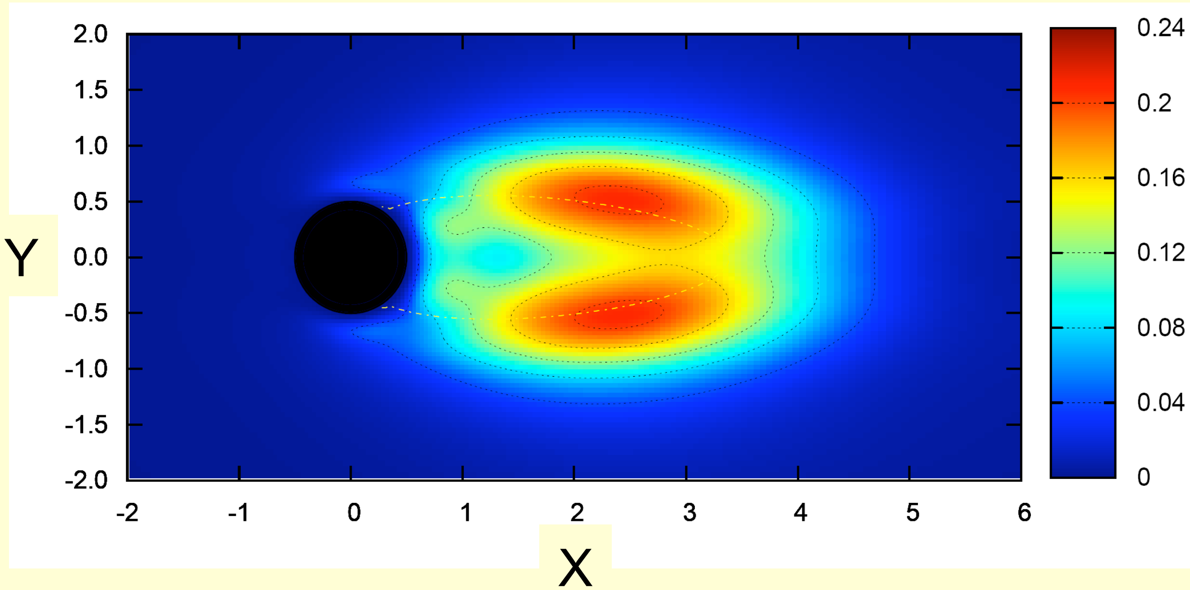
• Configuration: incompressible unconfined flow around a circular cylinder





Motivations

1) Giannetti & Luchini, JFM 2007: identification of the core (*wavemaker*) of the first instability of the cylinder wake (onset vortex shedding) by using the properties of the direct and adjoint unstable global modes



Sensitivity map of the eigenv. to spatially localized feedbacks at Re=50

Instability localized behind the cylinder !

2) Barkley, Phys. Rev. E (2005): by repeating the stability analysis on smaller and smaller subdomains he showed that only a small region behind the cylinder is responsible for 3D linear instability although the linear modes extend many cylinder diameters downstream the cylinder

Objectives and Outlines

- ✓ Generalize the method proposed by Giannetti & Luchini, (JFM 2007) in order to treat time-periodic base flows
- ✓ Apply the resulting approach to locate the *core* of the 3D instability in the wake of a circular cylinder
- ✓ Explain the results obtained by Barkley Phys. Rev. E (2005)
- ✓ Investigate the nature of the 3D instability (Mode A and B) (work in progress!)

Stability Analysis (Floquet)

2-D periodic base flow with period T
(Karman vortex shedding)

Small amplitude perturbation
(Fourier in the spanwise dir. z)

Flow decomposition

$$\mathbf{U}(x,y,z,t) = \mathbf{U}_b(x,y,t) + \epsilon \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{u}(x,y,\kappa,t) \ \exp(\mathrm{i}\kappa z) \mathrm{d}\kappa$$

$$P(x,y,z,t) = P_b(x,y,t) + \epsilon \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(x,y,\kappa,t) \ \exp(\mathrm{i}\kappa z) \mathrm{d}\kappa$$

Perturbation

$$\mathbf{u}(x,y,\kappa,t) = \hat{\mathbf{u}}(x,y,\kappa,t) \ \exp(\sigma t)$$

$$p(x,y,\kappa,t) = \hat{p}(x,y,\kappa,t) \ \exp(\sigma t)$$

Eigenvalue problem

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + \sigma \hat{\mathbf{u}} + \mathbf{L}_\kappa\{\mathbf{U}_b, Re\}\hat{\mathbf{u}} + \nabla \hat{p} = \mathbf{0}$$

$$\nabla_\kappa \cdot \hat{\mathbf{u}} = 0$$

+

$$\hat{\mathbf{u}}(t+T) = \hat{\mathbf{u}}(t)$$

$$\hat{p}(t+T) = \hat{p}(t)$$

=exp(Multiplier abilit (0)

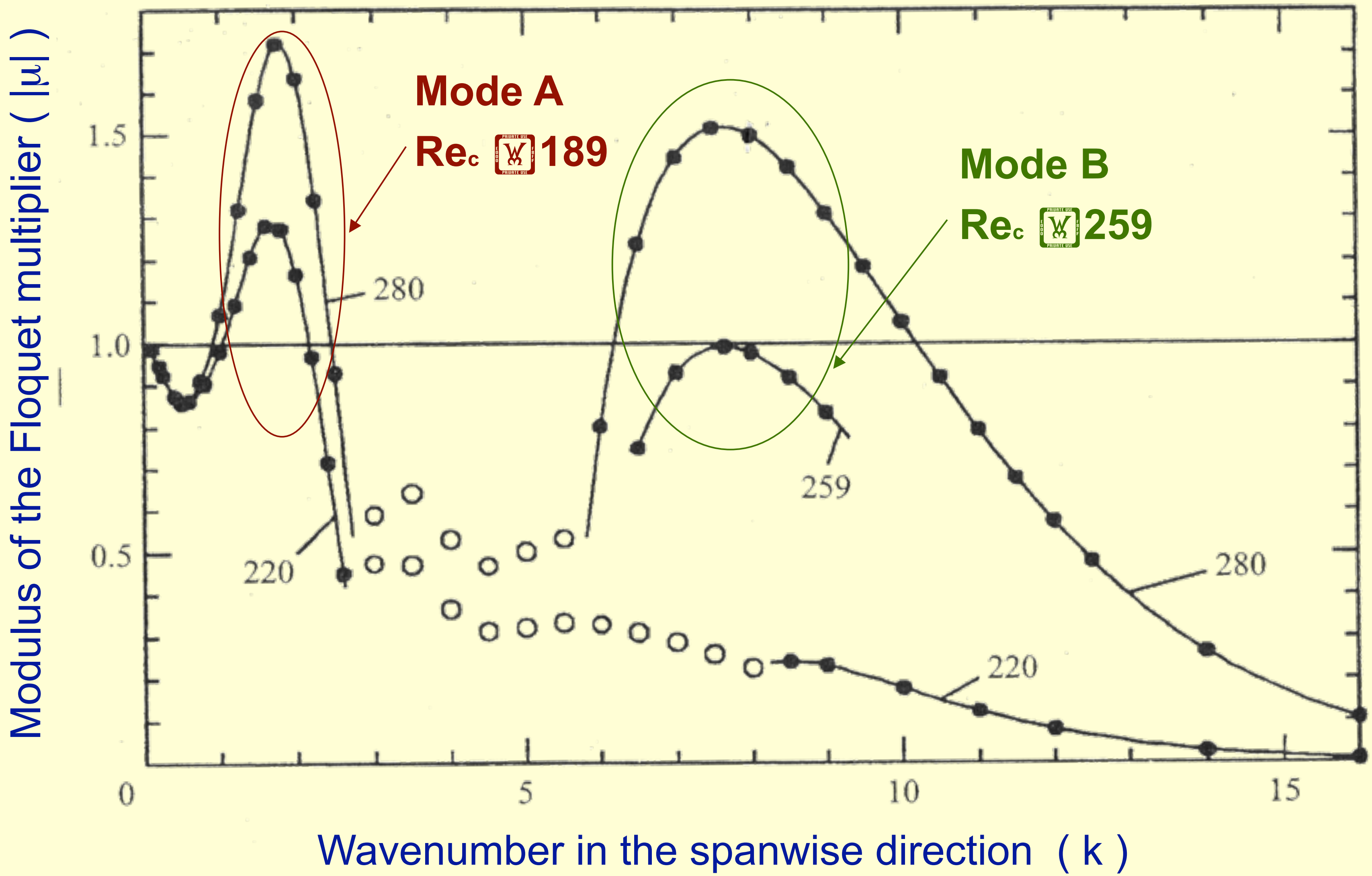
Definitions:

$$\mathbf{L}_\kappa\{\mathbf{U}_b, Re\}\mathbf{u} = \mathbf{U}_b \cdot \nabla_\kappa \mathbf{u} + \mathbf{u} \cdot \nabla_\kappa \mathbf{U}_b - \frac{1}{Re} \boldsymbol{\Delta}_\kappa \mathbf{u} \qquad \nabla_\kappa \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \mathrm{i}\kappa) \qquad \boldsymbol{\Delta}_\kappa \equiv \nabla_\kappa \cdot \nabla_\kappa$$

Exponent

(T period of the base flow) (0)

Floquet analysis (data from Barkley & Henderson, JFM 1996)



Floquet modes: characteristic symmetries

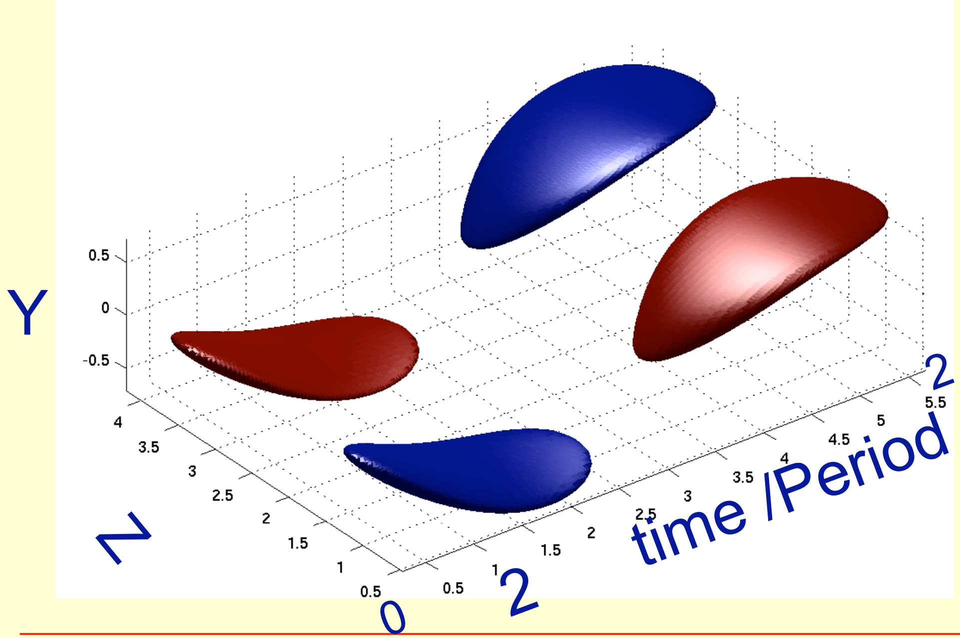
Mode A: symmetries:

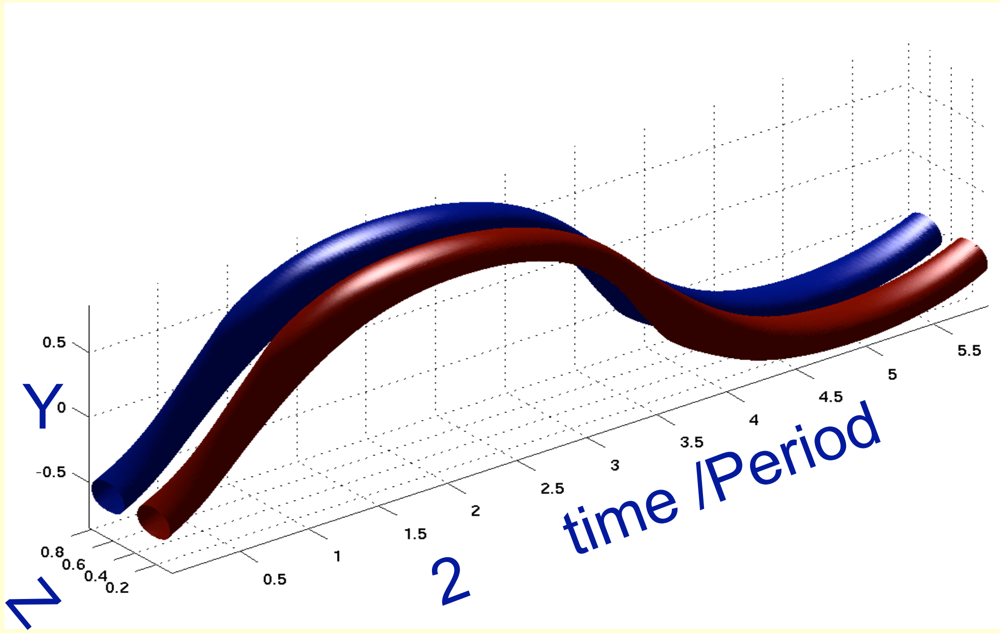
$$\hat{u}(x,y,\kappa,t) = \hat{u}(x,-y,t+T/2)$$
$$\hat{v}(x,y,\kappa,t) = -\hat{v}(x,-y,t+T/2)$$
$$\hat{w}(x,y,\kappa,t) = \hat{w}(x,-y,t+T/2)$$
$$\hat{\omega}_x(x,y,\kappa,t) = -\hat{\omega_x}(x,-y,t+T/2)$$

Mode B:symmetries:

$$\hat{u}(x,y,\kappa,t) = -\hat{u}(x,-y,t+T/2)$$
$$\hat{v}(x,y,\kappa,t) = \hat{v}(x,-y,t+T/2)$$
$$\hat{w}(x,y,\kappa,t) = -\hat{w}(x,-y,t+T/2)$$
$$\hat{\omega}_x(x,y,\kappa,t) = \hat{\omega_x}(x,-y,t+T/2)$$

Axial vorticity passing in time through a vertical plane two diameters downstream the cylinder





Structural stability analysis (1)

✓ Perturb the original eig. problem

$$\frac{\partial \hat{\mathbf{u}}'}{\partial t} + \sigma' \hat{\mathbf{u}}' + \mathbf{L}_\kappa\{\mathbf{U}_b, Re\}\hat{\mathbf{u}}' + \nabla_\kappa \hat{p}' = \delta\mathbf{H}(\hat{\mathbf{u}}', \hat{p}')$$

$$\nabla_\kappa \cdot \hat{\mathbf{u}}' = \delta R(\hat{\mathbf{u}}', \hat{p}')$$

Linear operators

Structural perturbations

✓ Expand around the unperturbed quantities

$$\hat{\mathbf{u}}' = \hat{\mathbf{u}} + \delta\hat{\mathbf{u}}; \qquad \hat{p}' = \hat{p} + \delta\hat{p}'; \qquad \sigma' = \sigma + \delta\sigma$$

Insert and linearize

✓ Equations for the *eigenvector drift and the eigenvalue drift*

$$\frac{\partial \delta\hat{\mathbf{u}}}{\partial t} + \sigma\, \delta\hat{\mathbf{u}} + \mathbf{L}_\kappa\{\mathbf{U}_b, Re\}\, \delta\hat{\mathbf{u}} + \nabla_\kappa\, \delta\hat{p} = -\delta\sigma\, \hat{\mathbf{u}} + \delta\mathbf{H}(\hat{\mathbf{u}}, \hat{p})$$

$$\nabla_\kappa \cdot\, \delta\hat{\mathbf{u}} = \delta R(\hat{\mathbf{u}}, \hat{p})$$

Known forcing terms

Structural stability analysis (2)

$$\frac{\partial \delta \hat{\mathbf{u}}}{\partial t} + \sigma\, \delta \hat{\mathbf{u}} + \mathbf{L}_\kappa \{\mathbf{U}_b, Re\}\, \delta \hat{\mathbf{u}} + \nabla_\kappa\, \delta \hat{p} = -\delta\sigma\, \hat{\mathbf{u}} + \delta \mathbf{H}(\hat{\mathbf{u}}, \hat{p})$$
$$\nabla_\kappa \cdot\, \delta \hat{\mathbf{u}} = \delta R(\hat{\mathbf{u}}, \hat{p}) \qquad (1)$$

- If we just wanted to determine the variation of the Floquet exponent for specific form of the structural perturbation we could solve the problem as stated above; but we can obtain a *much more powerful result*, *i.e.* the sensitivity of the Floquet exponent to an *arbitrary structural* perturbation, with the aid of *adjoint equations.*

- Key to this approach is to realize that the forced equations (1) only has a *periodic solution* if a compatibility condition is satisfied

- This can be derived through the application of the generalized Green's identity to (1)

Generalized Green's Identity and Adjoint Equations

$$\int_t^{t+T} \int_{\mathcal{D}} \left[-\delta\sigma\ \hat{\mathbf{u}} + \delta\mathbf{H}(\hat{\mathbf{u}}, \hat{p}) \right] \cdot \hat{\mathbf{f}}^+ + \left[\delta R(\hat{\mathbf{u}}, \hat{p}) \right]\ \hat{m}^+\ \mathrm{d}^2\mathbf{x}\ \mathrm{d}t = 0$$

Compatibility conditions

Sensitivity to spatially localized feedbacks

Assume structural perturbations in the form of a spatially localized force-velocity feedback

$$\frac{\partial \hat{\mathbf{u}}'}{\partial t} + \sigma'\hat{\mathbf{u}}' + \mathbf{L}_\kappa\{\mathbf{U}_b, Re\}\hat{\mathbf{u}}' + \nabla_\kappa \hat{p}' = \delta\mathbf{H}(\hat{\mathbf{u}}', \hat{p}')$$

$$\nabla_\kappa \cdot \hat{\mathbf{u}}' = \delta R(\hat{\mathbf{u}}', \hat{p}')$$

$$\delta\mathbf{H}(\mathbf{u}, p) = \delta(\mathbf{x} - \mathbf{x}_0)\ \mathbf{C}_0(\kappa) \cdot \mathbf{u}$$

$$\delta R(\mathbf{u}, p) = 0$$

Variation of the Floquet exponent due to the structural perturbations

Compatibility conditions with adjoint modes

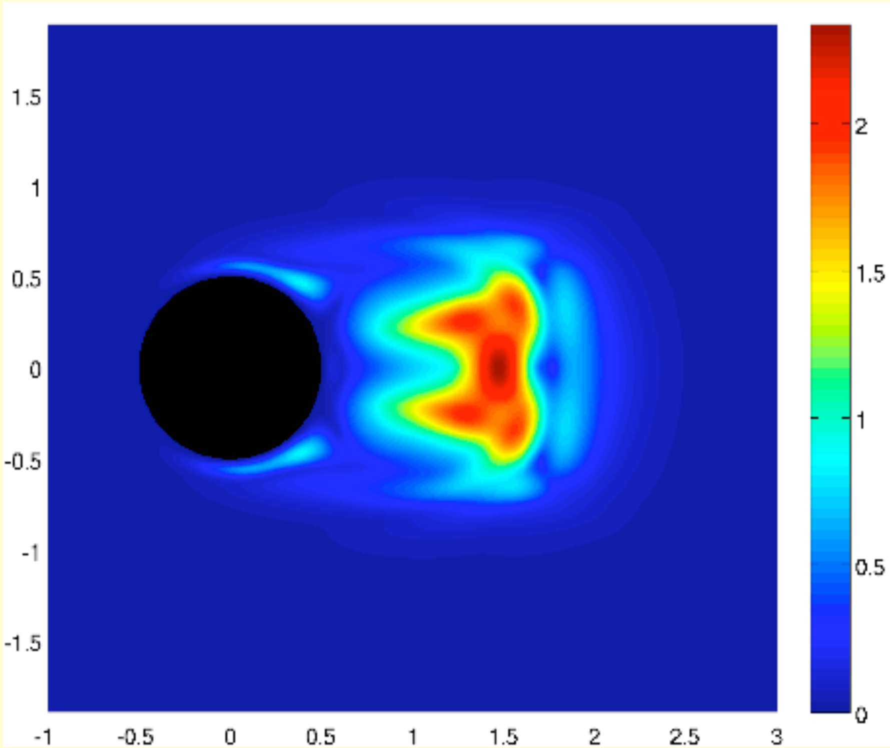
$$\delta\sigma = \frac{\int_t^{t+T} \int_D \hat{\mathbf{f}}^+ \cdot \mathbf{C}(x,y) \cdot \hat{\mathbf{u}}\ \mathrm{d}^2\mathbf{x}\ \mathrm{d}t}{\int_t^{t+T} \int_{\mathcal{D}} \hat{\mathbf{f}}^+ \cdot \hat{\mathbf{u}}\ \mathrm{d}^2\mathbf{x}\ \mathrm{d}t} = \mathbf{S}(x_0, y_0) : \mathbf{C}_0$$

Define the Sensitivity Tensor

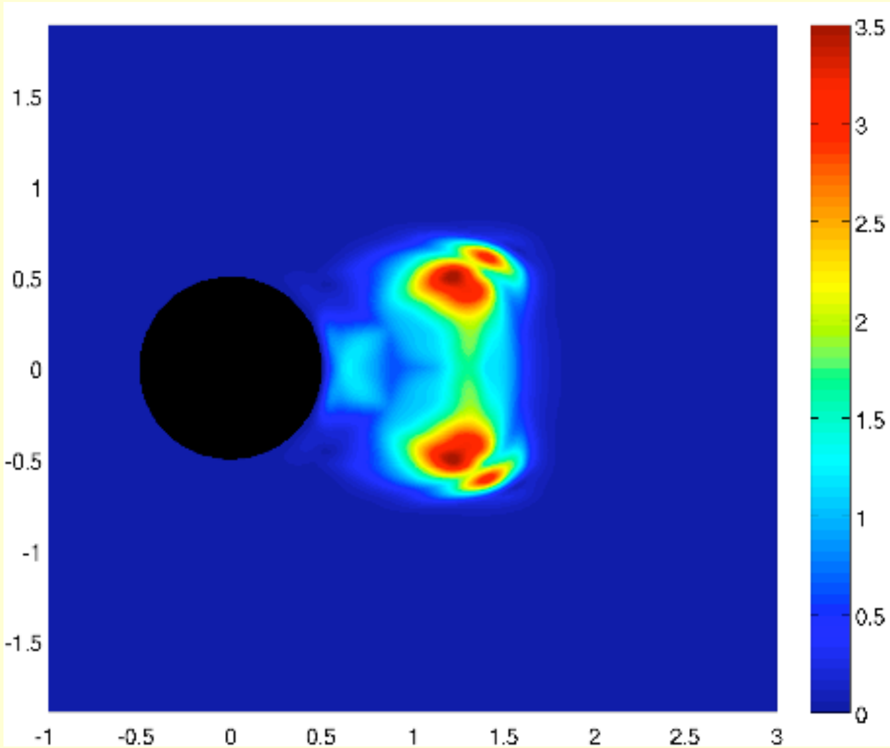
$$\mathbf{S}(x,y) = \frac{\delta\sigma}{\delta\mathbf{C}_0} = \frac{\int_t^{t+T} \hat{\mathbf{u}}(x,y,\kappa,t)\ \hat{\mathbf{f}}^+(x,y,\kappa,t)\ \mathrm{d}t}{\int_t^{t+T} \int_{\mathcal{D}} \hat{\mathbf{f}}^+ \cdot \hat{\mathbf{u}}\ \mathrm{d}^2\mathbf{x}\ \mathrm{d}t}$$

Sensitivity map of Floquet multiplier to spatially localized feddbacks

Mode A (Re=190, k=1.59)



Mode B (Re=260, k=7.64)

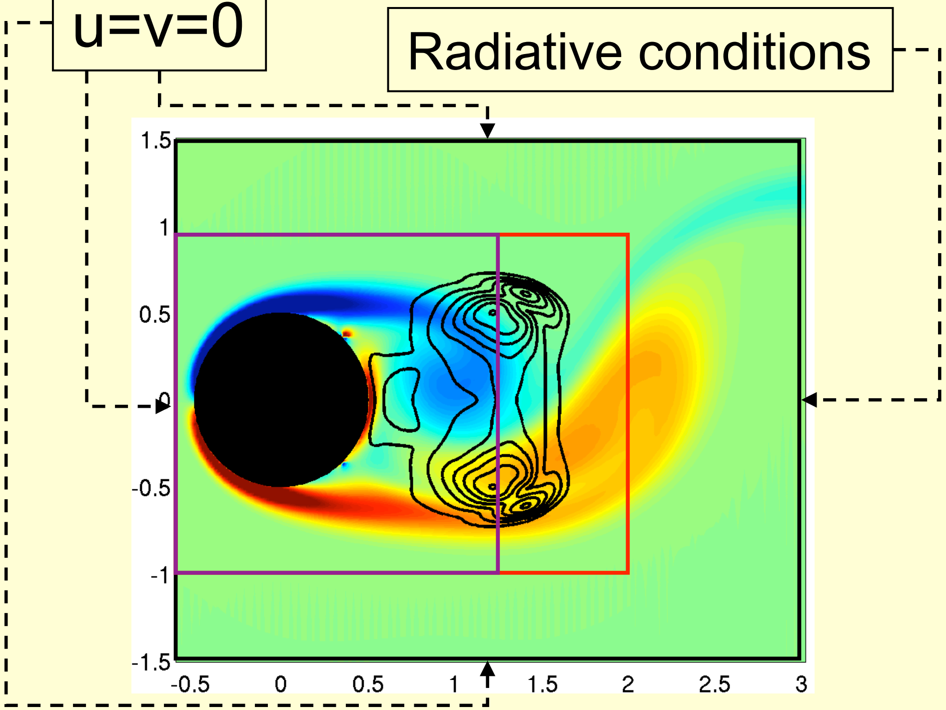


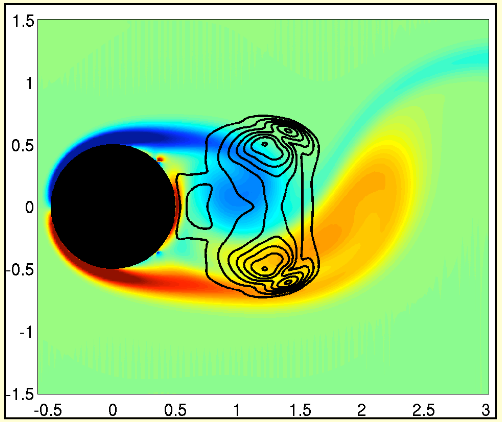
- Both for mode A and B the region of large sensitivity is localized just behind the cylinder where the vortices are still forming
- The characteristics of Mode A and B depend only on the details of the baseflow in the region of high sensitivity

Verification of the sensitivity map (mode B)

Stability analyses are carried out on progressively smaller subdomains and results are compared to those obtained using the whole domain

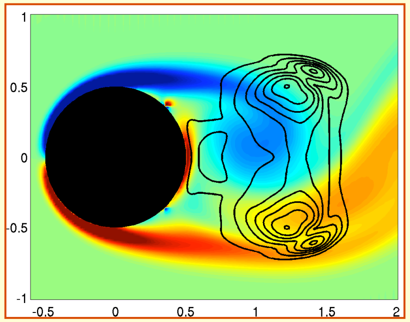
Variation of the Floquet multiplier:
(Re=260, K=7.64)
□ = 1.0737





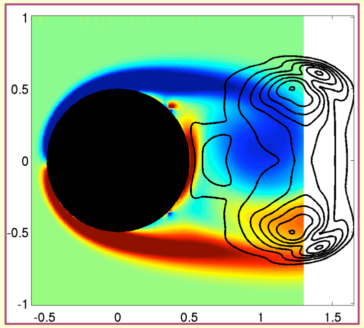
□=1.0736

- 0.01 □



□=1.0805

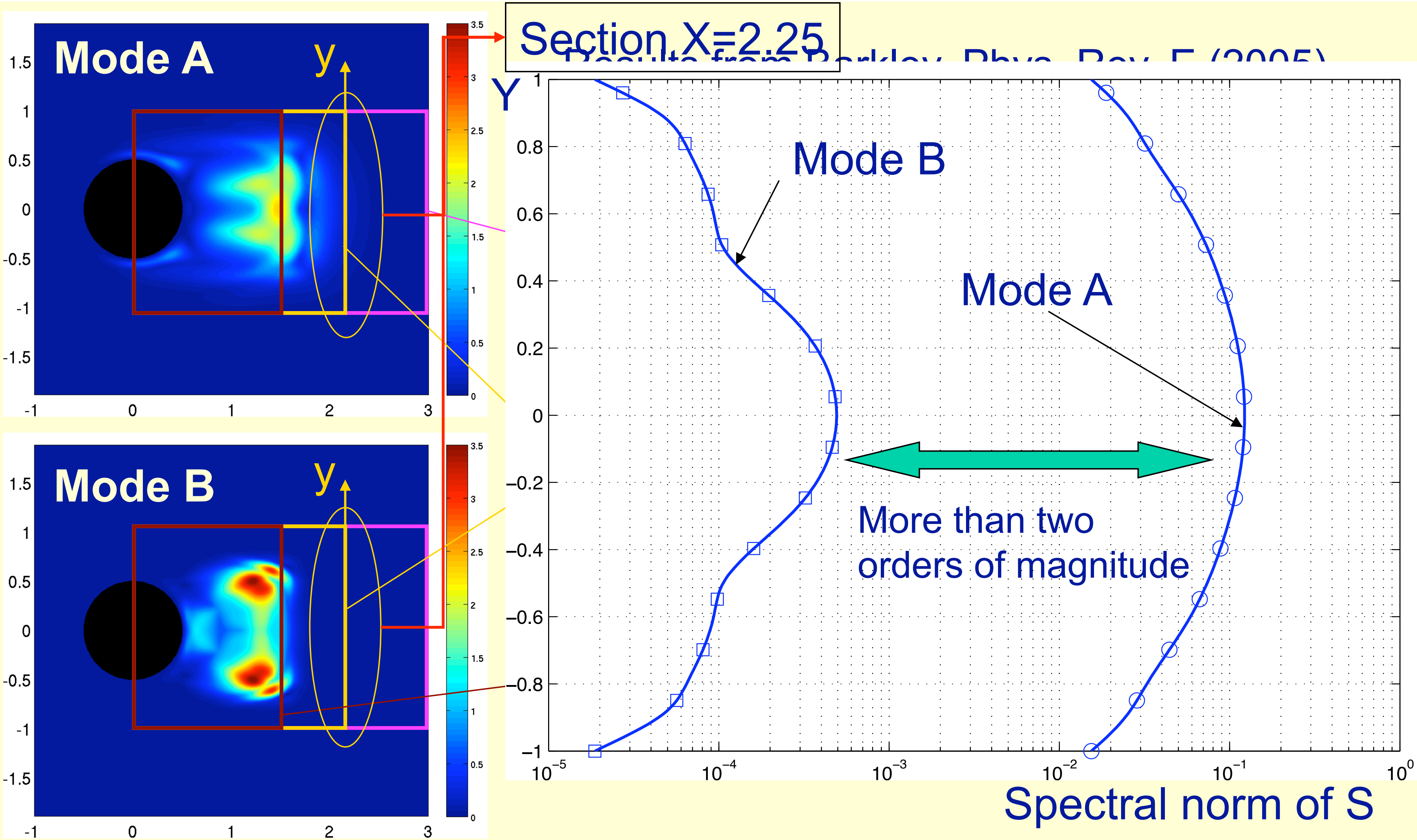
+ 0.63 □



□=16.384
9

+ 1390 □

Sensitivity maps and results from Barkley P.R.E. 2005



Recovering phase information (1)

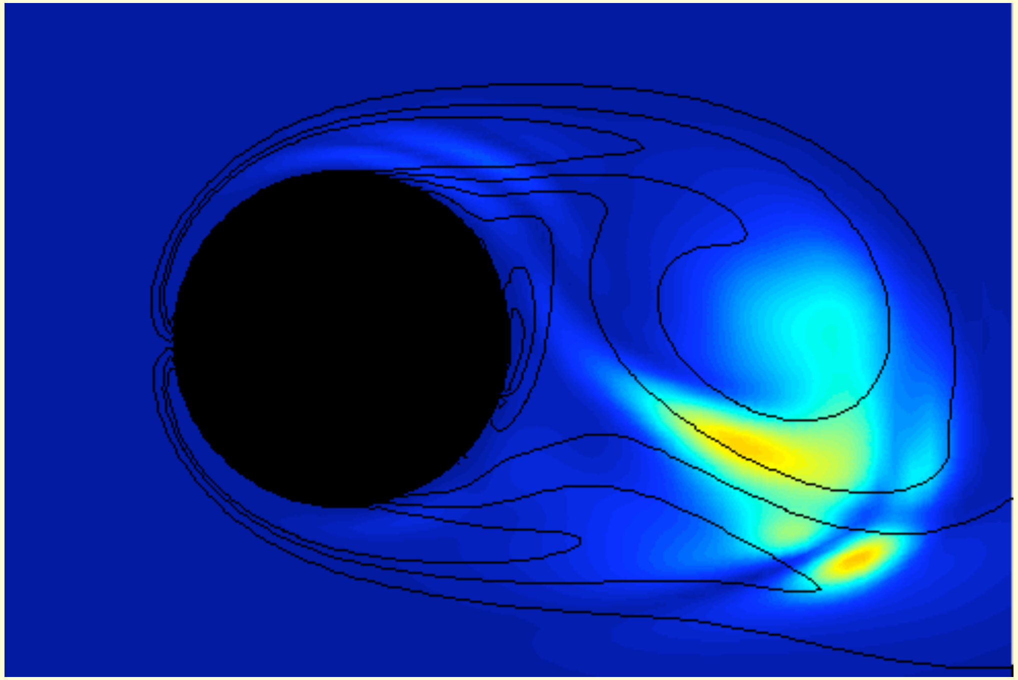
It is possible to recover the time evoltution of the sensitivity by looking over a period at feedbacks localized in space and in time.

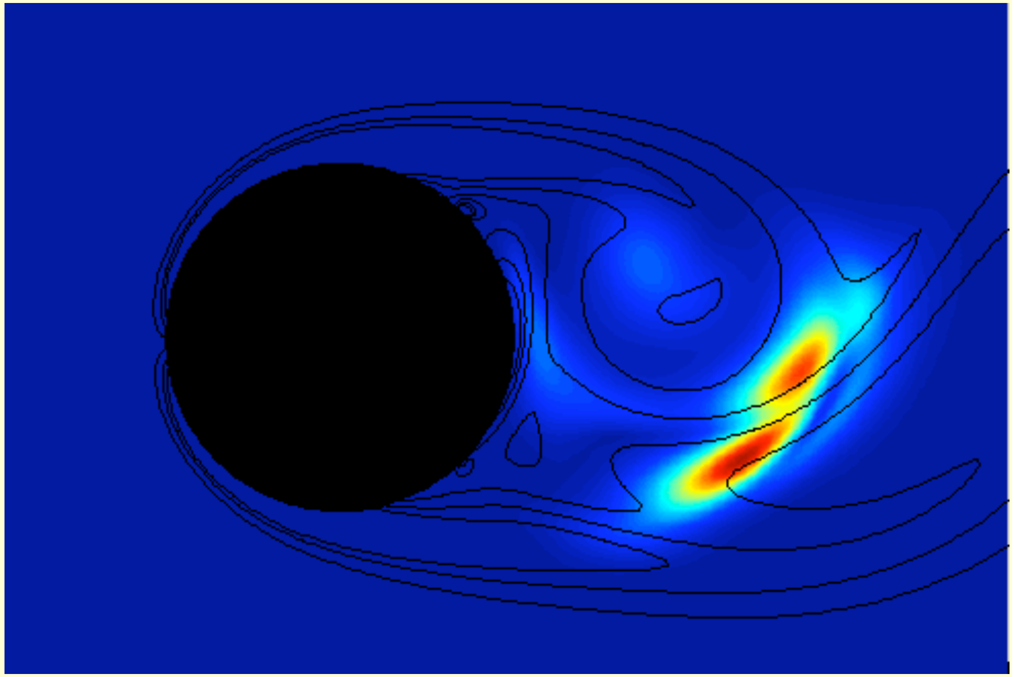
$$\delta \mathbf{H} \equiv \delta(t - t_0)\, \delta(x - x_0, y - y_0)\, \mathbf{C}_0(\kappa) \cdot \hat{\mathbf{u}}$$

$$\delta R \equiv 0$$

$$\delta\sigma = \frac{\displaystyle\int_t^{t+T} \int_D \hat{\mathbf{f}}^+ \cdot \delta\mathbf{H} \cdot \hat{\mathbf{u}} \, \mathrm{d}^2\mathbf{x} \, \mathrm{d}t}{\displaystyle\int_t^{t+T} \int_{\mathcal{D}} \hat{\mathbf{f}}^+ \cdot \hat{\mathbf{u}} \, \mathrm{d}^2\mathbf{x} \, \mathrm{d}t} = \mathbf{S}(x_0, y_0) : \mathbf{C}_0$$

Recovering phase information (2)





Mode A Re=190 k=1.58

Mode B Re=260 k=7.64

Time evolution of the sensitivity (spectral norm)

Local analysis (work in progress !!)

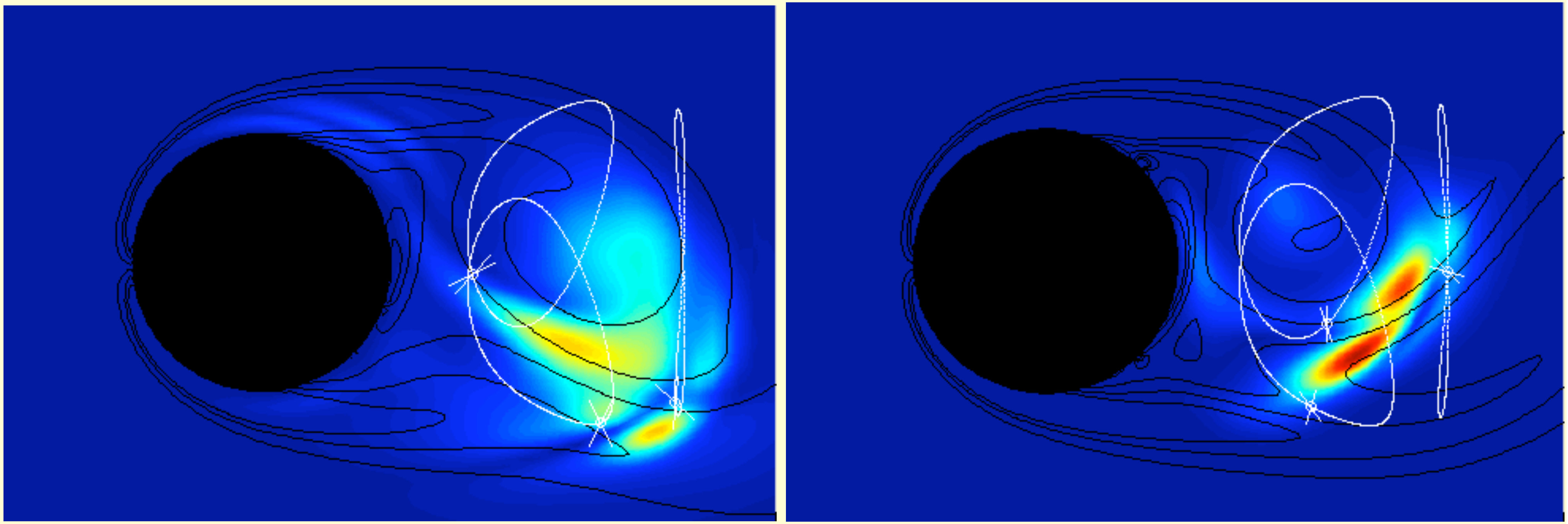
- The results obtained by recovering the phase information from our global analysis show that both for mode A and B the instability evolves in time in a complex way.

- The sharp localization of the instability core, however, suggests the possibility to *investigate the nature of mode A and B* through a *"local analysis" using a WKBJ approximation similar to* [A. Lifschitz & E. Hameiri POF 1991].

- In the inviscid limit the short wave-length instability propagates along Lagrangian trajectories.

- A special role can thus be ascribed to closed Lagrangian trajectories where the wave can feed back on itself giving rise to a self excited oscillation

Closed Lagrangian trajectories in the wake of the cylinder



Mode A Re=190 k=1.58

Mode B Re=260 k=7.64

We found 3 orbits! They all lie in the region of maximum sensitivity!

Conclusions

1) The approach introduced by Giannetti & Luchini (JFM 2007) has been extended to treat 3D perturbations on a 2D time-periodic base flow

3) The sensitivity of the Floquet exponent to a spatially localized feedback has been determined for the secondary instability of the wake of a circular cylinder.

5) The analysis shows that both for mode A and B the instability is extremely localized in space and evolves in time in a complex way

7) A set of closed Lagrangian trajectories in the wake of the cylinder were numerically determined. The position of the these trajectories are close to the maximum of the structural sensitivity.

Work in Progress

Study the development of short wave-length instabilities on the periodic orbits (local approach) and quantitatively compare the results to those obtained by the global stability analysis