Low dimensional model for control of the Blasius boundary layer by balanced truncation

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in collaboration with

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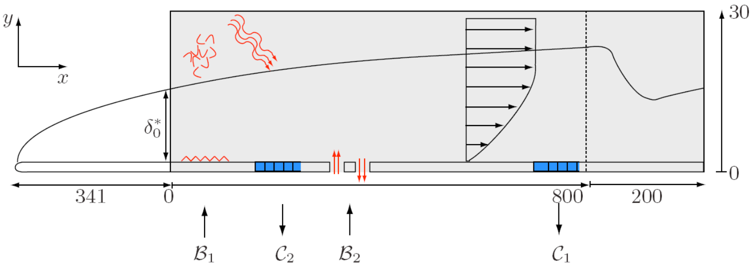
FLOW LINNÉ FLOW CENTRE

Outline of talk

- Flow instability, examining the initial value problem
- Adding inputs and outputs: The lifting procedure for wall actuation
- Reduced order models preserving input-output characteristics, balanced truncation
- LQG feedback control results based on reduced order model
- Conclusions

Flow set up and state space form



- Linearized Navier-Stokes equations

$$\dot{\mathbf{u}} = \mathcal{A}\mathbf{u}$$
$$\mathbf{u} = \mathbf{u}_0 \qquad \text{at} \qquad t = 0.$$

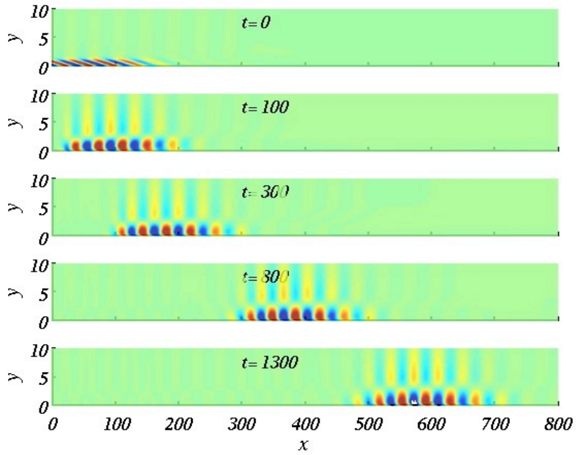
- Evolution operator central to both stability investigation and control design

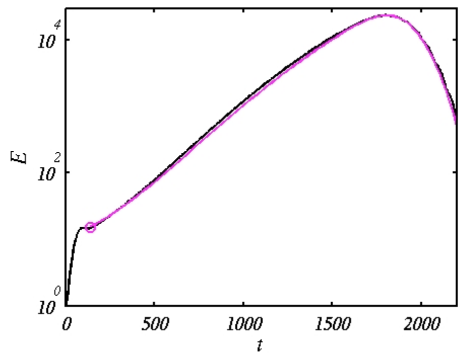
$$\mathbf{u}(t) = \mathcal{T}(t)\mathbf{u}_0 = \exp(\mathcal{A}t)\mathbf{u}_0$$

- Stability deals with full system

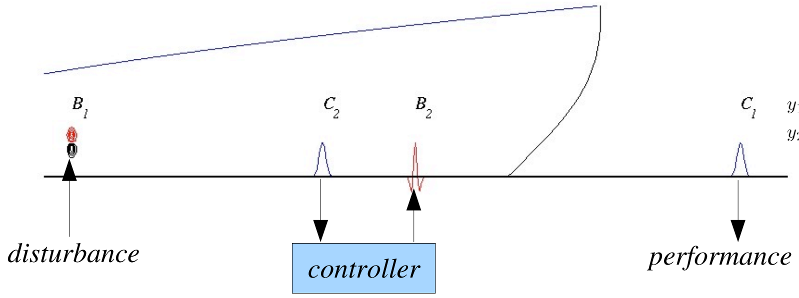
Non-modal growth

- Asymptotically stable; single eigenmodes are not observed
- Optimal growth computed from $\sigma\hat{\mathbf{u}} = \mathcal{T}^\dagger(t_o)\mathcal{T}(t_o)\hat{\mathbf{u}}$ yields wavepacket propagation
- Orr/TS mechanism





Adding inputs and outputs to the NS equations



$\mathcal{B}_1\psi$: External disturbances
$\mathcal{B}_2\phi$: Actuation
$y_1 = \mathcal{C}_1\mathbf{u}$: Sensor for feedback
$y_2 = \mathcal{C}_2\mathbf{u}$: Sensor for objective function

- Forced NS stokes equations with outputs

$$\dot{\mathbf{u}} = \mathcal{A}\mathbf{u} + \mathcal{B}_1\psi + \mathcal{B}_2\phi = \mathcal{A}\mathbf{u} + \mathcal{B}f \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \mathcal{C}_1\mathbf{u} \\ \mathcal{C}_2\mathbf{u} \end{bmatrix} = \mathcal{C}\mathbf{u}$$

- Input-output behaviour

$$y(t) = \mathcal{C}\mathcal{T}(t)\mathbf{u}_0 + \mathcal{C}\int_0^t \mathcal{T}(t-\tau)\mathcal{B}f(\tau)\mathrm{d}\tau$$

- For control design it is sufficient to capture the I-O behaviour

Obtaining a state space formulation for wall actuation

- Boundary controlled system not on state space form

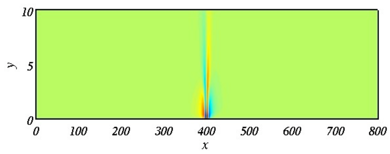
$$\dot{\mathbf{u}} = \mathcal{A}\mathbf{u}$$
$$\mathbf{u}(x,0,t) = \mathbf{u}_w\varphi(t) = (0, v_w(x))^T\varphi(t),$$

- Lifting procedure to obtain a volume forced system (*Högberg et al. 2003*)

$$\mathbf{u} = \mathbf{u}_h + \mathcal{Z}\varphi$$

- Steady state solution

steady state solution from DNS



$$0 = \mathcal{A}\mathcal{Z}$$
$$\mathcal{Z}(x,0) = \mathbf{u}_w = (0, v_w(x))^T$$

- Form an augmented system $\hat{\mathbf{u}} = (\mathbf{u}_h, \varphi)^T$

$$\dot{\hat{\mathbf{u}}} = \hat{\mathcal{A}}\hat{\mathbf{u}} + \hat{\mathcal{B}}_2\phi, \quad \text{with} \quad \hat{\mathcal{A}} = \begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix}, \hat{\mathcal{B}}_2 = \begin{bmatrix} -\mathcal{Z} \\ 1 \end{bmatrix}, \phi = \dot{\varphi}$$

Reduced order model for control

- Full system too big for optimization problem and online time integration

$$\dot{\mathbf{u}} = \mathcal{A}\mathbf{u} + \mathcal{B}f \qquad n > 10^5$$
$$y = \mathcal{C}\mathbf{u}$$

- Approximate by reduced system

$$\dot{\kappa} = \hat{A}\kappa + \hat{B}f$$
$$\hat{y} = \hat{C}\kappa \qquad m < 100$$

- Such that the IO is preserved

$$\|y - \hat{y}\| < \epsilon$$

- One systematic approach is balanced truncation (*Moore 81*)

Controllability and observability

- How do the inputs affect the outputs?
- Generated states from input (controllability)

$$\mathbf{u}_0 = \int_{-\infty}^{0} \mathcal{T}(-\tau)\mathcal{B}f(\tau)\mathrm{d}t = \mathcal{L}_c f$$

- Generated signal from states (observability)

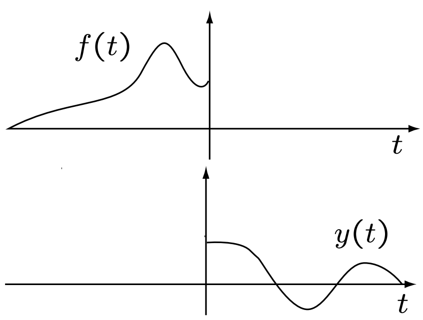
$$y(t) = \mathcal{C}\mathcal{T}(t)\mathbf{u}_0 = \mathcal{L}_o(t)\mathbf{u}_0$$

- Hankel operator

$$y(t) = \mathcal{L}_o(t)\mathcal{L}_c f(t) = (\mathcal{H}f)(t)$$

- Gramians

$$\mathcal{P} = \mathcal{L}_c\mathcal{L}_c^\dagger = \int_0^\infty \mathcal{T}(\tau)\mathcal{B}\mathcal{B}^\dagger\mathcal{T}^\dagger(\tau)\,\mathrm{d}\tau$$
$$\mathcal{Q} = \mathcal{L}_o^\dagger\mathcal{L}_o = \int_0^\infty \mathcal{T}^\dagger(\tau)\mathcal{C}^\dagger\mathcal{C}\mathcal{T}(\tau)\,\mathrm{d}\tau$$



Gramians

- Approximate integrals by quadrature (snapshots)

$$\mathcal{P} = \int_0^\infty \mathcal{T}(\tau)\mathcal{B}\mathcal{B}^\dagger\mathcal{T}^\dagger(\tau)\,\mathrm{d}\tau \approx XX^T, \quad X = [\mathcal{B}, \exp(\mathcal{A}t_1)\mathcal{B}, \ldots, \exp(\mathcal{A}t_n)\mathcal{B}]$$

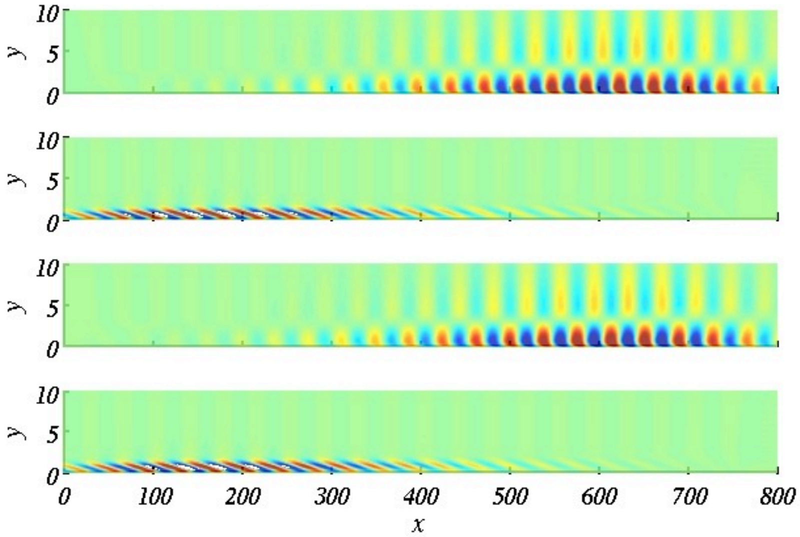
$$\mathcal{Q} = \int_0^\infty \mathcal{T}^\dagger(\tau)\mathcal{C}^\dagger\mathcal{C}\mathcal{T}(\tau)\,\mathrm{d}\tau \approx YY^T, \quad Y = [\exp(\mathcal{A}^\dagger t_m)\mathcal{C}^\dagger, \ldots, \exp(\mathcal{A}^\dagger t_1)\mathcal{C}^\dagger, \mathcal{C}^\dagger]$$

- POD modes $\mathrm{eig}\{XX^T\}$
  - Ranks states most easily influenced by input
- Adjoint POD modes $\mathrm{eig}\{YY^T\}$
  - Ranks states most easily sensed by output
- Balanced modes mutually diagonalize Gramians

$$\mathcal{P}\mathcal{Q}\mathbf{v} = \sigma\mathbf{v} \quad \Rightarrow \quad \underbrace{X^TYY^TX}_{m\times m}\,a = \sigma a, \quad \mathbf{v} = Xa$$

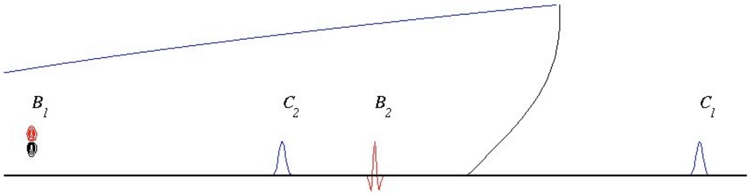
Balanced modes

- Direct and adjoint balanced modes

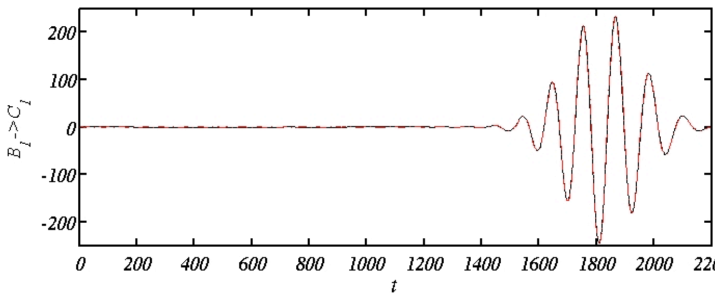


- Use these for oblique Galerkin projection to create reduced order model

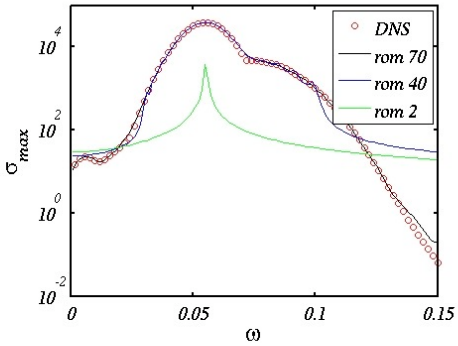
Preserving input-output behaviour



Impulse response from disturbance to objective function



Frequency response from all inputs to all outputs



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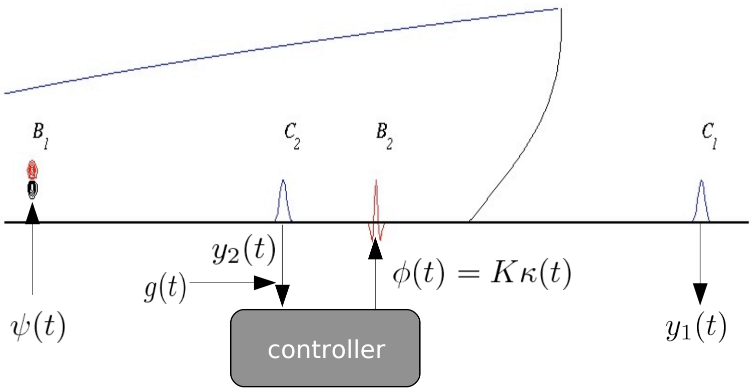
Optimal Feedback Control – LQG



Find an optimal control signal $\phi(t)$ based on the measurements $y_2(t)$ such that in the presence of external disturbances $\psi(t)$ and measurement noise $g(t)$ the output $y_1(t)$ is minimized.

→ Solution: LQG/H2

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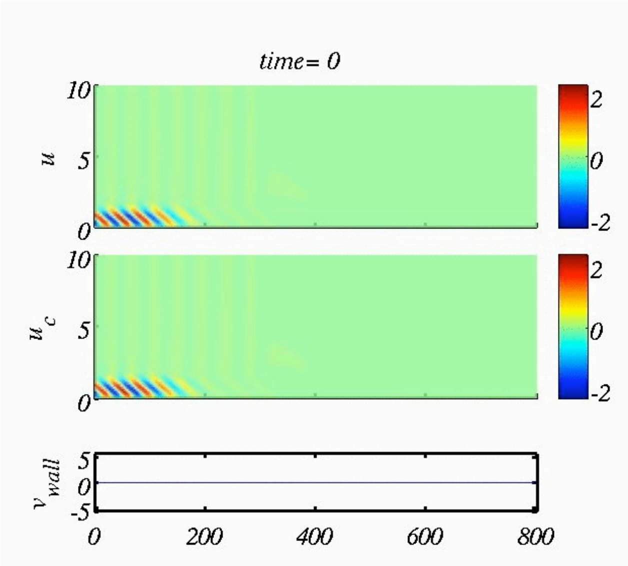
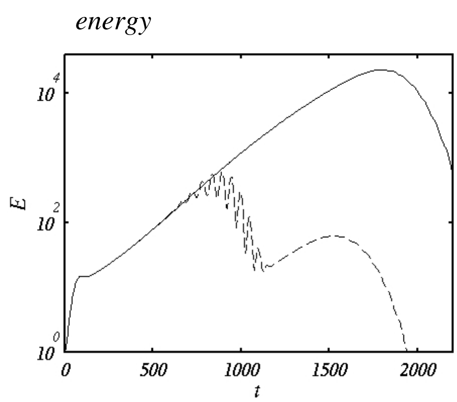
Control of the propagating wavepacket

- Small estimator running online
- Control signal fed into actuator in DNS





Conclusions

- Balanced modes give low order models preserving input-output relationship between sensors and actuators
- Feedback control of Blasius flow
  - Impulse and frequency response well captures
  - Reduced order models with balanced modes used in LQG control
  - Controller based on small number of modes works well in DNS
- Framework enables LQG control for many complex flows
  - DNS/ADNS is all that is needed