Optimal feedback control applied to stability and turbulence

P. Luchini

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Outline

Optimal feedback control applied to stability and turbulence
P. Luchini
Control theory
State representation
Input-output representation
Choice of the objective function
Application to stability
Stabilizing a wake
Application to Turbulence
The mean linear response
A control-kernel example
Conclusions

What is special about stability and turbulence

as control problems?

Traditional control problem	Stability & turbulence
Mechanical guidance or industrial process	Flow of a continuum
O.D.E.	P.D.E.

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$\Downarrow$	$\Downarrow$
Kalman filter, Riccati equation.	Reduced-order models, iterative eigenvalue methods, input-output formulation.

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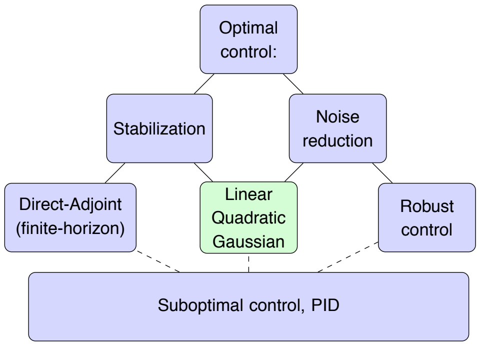
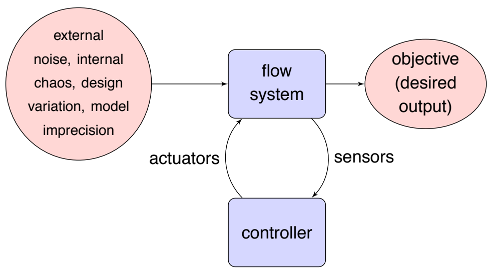
Control tree



The basics

Optimal control:
designing a feedback controller through optimization of a suitable objective function.



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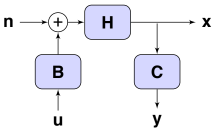
Two flavours

Note:
noise input and actuator input are different;
desired output and sensor output are different.



LQG optimal control

For the desired output of a Linear system, the statistical expectation of a Quadratic norm is minimized in the presence of stochastic Gaussian noise of known correlation function.

Robust control

For the desired output the maximum of a quadratic norm is minimized over the set of noises of a separately specified input norm.

Two flavours

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LQG optimal control

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Static vs. dynamic optimization

Static optimization of a function $E(\mathbf{x})$:

- Bracketing methods: bisection, simplex, etc.
- Gradient-based methods: find a zero of the gradient. If E is quadratic, the gradient is a linear system: direct methods, iterative methods.

Static vs. dynamic optimization

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Dynamic optimization of a functional $E[x(t)]$:

reduces to above if $x(t)$ is discretized and E is treated as a function of the vector $\mathbf{x}$ of time samples. Adjoint equations provide the gradient.

However causality is involved: the present state of any dynamical system and of its physically realizable controller cannot depend on the future and cannot affect the past.

Statistics of a static linear system

Given:

- a stochastic (vector) variable n_j;
- a linear function $x_i = H_{ij}n_j$ and its adjoint $H^+_{ji} = H^*_{ij}$;
- a quadratic form $E = x^*_i Q_{ij} x_j$;
- a correlation matrix $N_{ij} = \langle n_i n^*_j \rangle$

we find:

$$\langle E \rangle = \langle x^*_i Q_{ij} x_j \rangle = Q_{ij} H_{jh} N_{hk} H^*_{ik} =$$

$$= \underbrace{\mathrm{Tr}(\mathbf{QHNH}^+) = \mathrm{Tr}(\mathbf{H}^+\mathbf{QHN}) = \mathrm{Tr}(\mathbf{NH}^+\mathbf{QH}) = \mathrm{Tr}(\mathbf{HNH}^+\mathbf{Q})}_{\text{circular product}}$$

Duality: $\langle E \rangle$ is left unchanged if $\mathbf{Q} \rightleftharpoons \mathbf{N}$ and $\mathbf{H} \rightleftharpoons \mathbf{H}^+$

With control:

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system definition becomes

$x_i = H_{ij}(n_j + B_{jh}u_h)$	noise input (random) + actuator input (given)
$u_h = K_{hk}y_k$	controller (kernel to be determined)
$y_k = C_{kl}H_{lm}n_m$	sensor output (given)
equivalent to:	$\mathbf{H} \Leftarrow \mathbf{H} + \mathbf{HBKCH}$

$$\langle E \rangle = \mathrm{Tr}[\mathbf{Q}(\mathbf{H}+\mathbf{HBKCH})\mathbf{N}(\mathbf{H}+\mathbf{HBKCH})^+]$$

Quadratic objective $\Rightarrow$ linear equation for the kernel

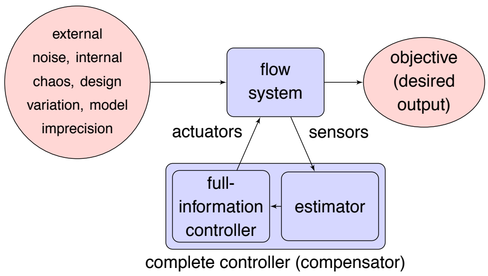
$$\frac{1}{2}\frac{\mathrm{d}\langle E \rangle}{\mathrm{d}\mathbf{K}^+} = \mathbf{B}^+\mathbf{H}^+\mathbf{Q}(\mathbf{H}+\mathbf{HBKCH})\mathbf{NH}^+\mathbf{C}^+ = 0.$$

$$\mathbf{K} = -(\mathbf{B}^+\mathbf{H}^+\mathbf{QHB})^{-1}\mathbf{B}^+\mathbf{H}^+\mathbf{QHNH}^+\mathbf{C}^+(\mathbf{CHNH}^+\mathbf{C}^+)^{-1}$$

Note: $\mathbf{B}^+\mathbf{H}^+\mathbf{QHB}$ is what a direct-adjoint computation gives.

Separation principle

$$\mathbf{K} = -\underbrace{(\mathbf{B}^+\mathbf{H}^+\mathbf{QHB})^{-1}\mathbf{B}^+\mathbf{H}^+\mathbf{Q}}_{\text{full-information controller}}\underbrace{\mathbf{HNH}^+\mathbf{C}^+(\mathbf{CHNH}^+\mathbf{C}^+)^{-1}}_{\text{estimator}}$$



Feedback vs. feedforward

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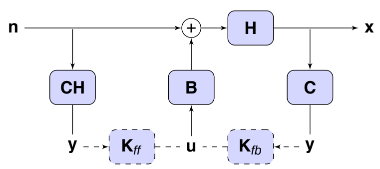
Feedforward: sensors sense input noise

$$\mathbf{y} = \mathbf{CHn}$$

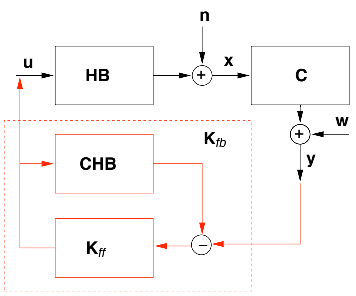


Feedback: sensors sense controlled output (state)

$$\mathbf{y} = \mathbf{Cx} = \mathbf{CH}(\mathbf{n} + \mathbf{BKy})$$

Conversion feedback ↔ feedforward

$$\mathbf{K}_{fb} = (\mathbf{1} + \mathbf{K}_{ff}\mathbf{CHB})^{-1}\mathbf{K}_{ff}$$



Statistics of a discrete dynamical system

(discrete stochastic process)

If the process is discrete in time x_i, n_j are interpreted as vectors of time samples,

- $x_i = H_{ij}n_j$ becomes a dynamical system with impulse response H_{ij} (causality requires $H_{ij} = 0$ for $i < j$);
- $E = x_i^* Q_{ij} x_j$ becomes a sum over discrete time;
- $N_{ij} = \langle n_i n_j^* \rangle$ becomes a time autocorrelation.

Remains true that:

$$\langle E \rangle = \mathrm{Tr}[\mathbf{Q}(\mathbf{H} + \mathbf{HBKCH})\mathbf{N}(\mathbf{H} + \mathbf{HBKCH})^{+}]$$

$$\mathbf{K} = (\text{full-information controller matrix}) \times (\text{estimator matrix})$$

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but causality of **K** must now be enforced.

The *causal inverse*

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$$\mathbf{X} \stackrel{\text{def}}{=} \mathbf{A}^{c.i.}\mathbf{B} \quad : \quad \begin{cases} A_{ij} X_{jk} = B_{ik} & \text{for} \quad i \geq k \\ X_{ik} = 0 & \text{for} \quad i < k \end{cases} \tag{1}$$

With this definition:

$$\mathbf{K} = -(\mathbf{B}^+\mathbf{H}^+\mathbf{QHB})^{c.i.}\mathbf{B}^+\mathbf{H}^+\mathbf{QHNH}^+\mathbf{C}^+(\mathbf{CHNH}^+\mathbf{C}^+)^{c.i.}$$

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The Wiener filter and the Kalman filter are two smart ways to obtain the (continuous version of) causal inverse in a computationally efficient way.

Definition (1) is just another linear system, and can always be used for comparison.

Statistics of a continuous dynamical system

(continuous stochastic process)

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For a time-continuous process the sums become integrals.

- $\mathbf{x} = \int_{-\infty}^{t} \mathbf{H}(t,\tau)\mathbf{n}(\tau)\,\mathrm{d}\tau$ becomes a dynamical system with impulse response $\mathbf{H}(t,\tau)$ (where $\mathbf{H}(t,\tau) = 0$ when $t < \tau$ for causality);
- the quadratic form $E = \int_{-\infty}^{\infty} \mathbf{x}^{*}(t_1)\mathbf{Q}(t_1,t_2)\mathbf{x}(t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$;
- the correlation matrix $\mathbf{N}(\tau_2,\tau_1) = \langle \mathbf{n}(\tau_2)\mathbf{n}^{*}(\tau_1)\rangle$.
- Objective function:

$$\langle E\rangle = \int \mathrm{Tr}[\mathbf{Q}(t_1,t_2)\mathbf{H}(t_2,\tau_2)\mathbf{N}(\tau_2,\tau_1)\mathbf{H}^{+}(\tau_1,t_2)]\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1$$

Separation principle $\Rightarrow$ Separate control and estimation problems.
Causality $\Rightarrow$ Wiener or Kalman filter.
If time-invariant: discrete: $H_{ij} = H_{i-j}$ (Töplitz)
continuous: $H(T,\tau) = H(t-\tau)$ (convolution)

You Are Here:

The concept of state

Definition:
if **x** is rich enough that its knowledge at a single instant completely determined the system's future (for zero input) and subsumes the system's past we call **x** state.

In practice this happens when the description of the system is a difference or a differential equation in normal form.

$$
\begin{aligned}
\mathbf{x}_{n+1} &= \mathbf{A}\mathbf{x}_n + \mathbf{B}\mathbf{u}_n &&: \mathbf{H}_{ij} = \mathbf{A}^{(i-j)} \\
\mathrm{d}\mathbf{x}/\mathrm{d}t &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) &&: \mathbf{H}(t-\tau) = \mathrm{e}^{(t-\tau)\mathbf{A}} \\
\mathbf{y} &= \mathbf{C}\mathbf{x}
\end{aligned}
$$

This definition carries over to partial differential equations, provided **x** is the sufficient set of initial conditions.
Standard control theory is strongly biased towards ordinary differential equations.

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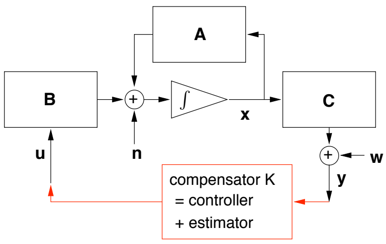
State representation of a dynamical system



Matrix **A** describes the system (*i.e.* linearized Navier-Stokes eqs.).
n is a white noise.

The linear optimal control problem and its solution

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The classical full-information control problem is formulated as follows: for the state **x** and the control **u** related via the *state equation*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \text{on} \quad 0 < t < T \qquad \text{with} \qquad \mathbf{x} = \mathbf{x}_0 \quad \text{at} \quad t = 0,$$

find the control **u** that minimizes the *cost function*

$$E = \frac{1}{2}\int_0^T [\mathbf{x}^*\mathbf{Q}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u}]\,\mathrm{d}t.$$

The *adjoint variable* **r** is introduced as a Lagrange multiplier.
Taking variations of the augmented cost function

$$E = \int_0^T \frac{1}{2}[\mathbf{x}^*\mathbf{Q}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u}] + \mathbf{r}^*[\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{u}]\,dt.$$

gives $\dot{\mathbf{r}} = -\mathbf{A}^+\mathbf{r} - \mathbf{Q}\mathbf{x}; \qquad \mathbf{R}\mathbf{u} = -\mathbf{B}^+\mathbf{r}; \qquad \mathbf{r} = 0 \quad \text{at } t = T.$

The boundary-value problem

The state and adjoint equations may be combined in matrix form

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{Z}\mathbf{z} \quad \text{where} \quad \mathbf{Z} = \mathbf{Z}_{2n\times 2n} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{+} \\ -\mathbf{Q} & -\mathbf{A}^{+} \end{bmatrix}, \qquad (2)$$

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{r} \end{bmatrix}, \quad \text{and} \quad \begin{cases} \mathbf{x} = \mathbf{x}_0 & \text{at } t = 0, \\ \mathbf{r} = 0 & \text{at } t = T. \end{cases}$$

(**Z** has a *Hamiltonian symmetry*, such that eigenvalues appear in pairs of equal imaginary and opposite real part.)
This linear ODE is a *two-point boundary value problem*. It may be transformed into an *initial-value problem* by assuming there exists a relationship between the state vector $\mathbf{x}(t)$ and adjoint vector $\mathbf{r}(t)$ via a matrix $\mathbf{X}(t)$ such that $\mathbf{r} = \mathbf{X}\mathbf{x}$, and inserting this solution ansatz into (2) to eliminate **r**.

The Riccati equation

It follows that matrix **X** obeys the *differential Riccati equation*

$$-\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{A}^{+}\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{+}\mathbf{X} + \mathbf{Q} \quad \text{where} \quad \mathbf{X}(T) = 0. \qquad (3)$$

Once **X** is known, the optimal value of **u** may then be written in the form of a *feedback control rule* such that

$$\mathbf{u} = \mathbf{K}\mathbf{x} \quad \text{where} \quad \mathbf{K} = -\mathbf{R}^{-1}\mathbf{B}^{+}\mathbf{X}.$$

Finally, if the system is time invariant and we take the limit that $T \to \infty$, the matrix **X** in (3) may be marched to steady state. This steady state solution for **X** satisfies the *continuous-time algebraic Riccati equation*

$$0 = \mathbf{A}^{+}\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{+}\mathbf{X} + \mathbf{Q},$$

where additionally **X** is constrained such that $\mathbf{A} + \mathbf{B}\mathbf{K}$ is stable.

How is the Riccati equation solved?

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by transforming it into a linear eigenvalue problem.
But we already had a linear problem to start with!

As a historical aside, the original transformation devised by *Jacopo Francesco Riccati* (1676 - 1754) was meant to solve a *nonlinear* first-order ordinary differential equation by transforming it into a *linear* second-order one; the above derivation just applied it the other way round.

How is the Riccati equation solved?

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As a historical aside, the original transformation devised by *Jacopo Francesco Riccati* (1676 - 1754) was meant to solve a *nonlinear* first-order ordinary differential equation by transforming it into a *linear* second-order one; the above derivation just applied it the other way round.

So, let's unwind the transformation...

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$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{r} \end{bmatrix}, \quad \text{and} \quad \begin{cases} \mathbf{x} = \mathbf{x}_0 & \text{at } t = 0, \\ \mathbf{r} = 0 & \text{at } t = T. \end{cases}$$

This linear ODE is a *two-point boundary value problem*.

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is in fact a classical eigenvalue problem.

The solution of a linear time-invariant system of O.D.E.'s is provided by its eigenvectors. Let the eigenvector decomposition of the $2n \times 2n$ matrix **Z** be

$$\mathbf{Z} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} \quad \text{where} \quad \mathbf{V} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{r} \end{bmatrix}$$

and the eigenvalues of **Z** appearing in the diagonal matrix $\mathbf{\Lambda}$ are enumerated in order of increasing real part. Since

$$\mathbf{z} = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}\mathbf{z}_0$$

the solutions **z** that obey the boundary conditions at $t = \infty$ are spanned by the first n columns of **V**. The direct (**x**) and adjoint (**r**) parts of these columns are related as $\mathbf{r} = \mathbf{X}\mathbf{x}$, where

$$\mathbf{X} = \mathbf{V}_{21}\mathbf{V}_{11}^{-1}.$$

Perspective

- Our group has experience in the computation and use of direct and adjoint modes of recirculating flows, linearized about unstable equilibria.
- Recent advances in multigrid numerical methods for this purpose have been presented at the 5th Symposium on Bluff Body Wakes and Vortex-Induced Vibrations (Dec 2007).
- A technique developed at UCSD, based solely on the unstable eigenvalues and corresponding left eigenvectors of the linearized open-loop system, provides the minimal-energy stabilizing controller and is a perfect match of the above eigenvalue algorithm. The results of this collaboration will be presented later in this conference.
- A multigrid solver for the first few eigenvalues and eigenvectors of the full **Z** matrix is under development.

You Are Here:

Size considerations

State representation

$$\mathbf{K} = -\underbrace{(\mathbf{B}^+\mathbf{H}^+\mathbf{QHB})^{c.i.}(\mathbf{B}^+\mathbf{H}^+\mathbf{Q})}_{\text{full-information controller}}\underbrace{(\mathbf{HNH}^+\mathbf{C}^+)(\mathbf{CHNH}^+\mathbf{C}^+)^{c.i.}}_{\text{estimator}}$$

When the sizes of actuator **u** and sensor **y** are much smaller than the size of state **x**, splitting the compensator into two Kalman-filter problems is no longer convenient.

Input-output representation

$$\mathbf{K} = -\underbrace{(\mathbf{B}^+\mathbf{H}^+\mathbf{QHB})^{c.i.}}_{2}\underbrace{(\mathbf{B}^+\mathbf{H}^+\mathbf{Q})(\mathbf{HNH}^+\mathbf{C}^+)}_{1}\underbrace{(\mathbf{CHNH}^+\mathbf{C}^+)^{c.i.}}_{3}$$

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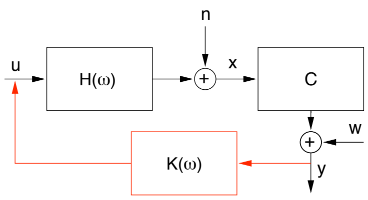
Fourier representation of the optimal controller



Problem: to find $K(\omega)$ such that $x^*Qx + u^*Ru$ is a minimum when $x = Hu + n$; $y = Cx + w$; $u = Ky$, w is measurement noise and n has the statistics of the actual, measured, turbulent-flow fluctuations.

After Fourier transformation:

space structure is trivial: wavenumbers decouple;

time dynamics is the true difficulty: causality has to be enforced.

The estimation problem: Wiener versus Kalman optimal filtering

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- **Kalman** (1960): find an instantaneous K such that $(x - \tilde{x})^* Q (x - \tilde{x})$ is a minimum with $dx/dt = Ax + KC\tilde{x}$; $d\tilde{x}/dt = A\tilde{x} + n$ and n a white gaussian noise.
- **Wiener** (1939): find a causal $K(\omega)$ such that $(x - \tilde{x})^* Q (x - \tilde{x})$ is a minimum with $x = KC(\tilde{x} + n)$, where $\tilde{x}$ is a signal and n a noise, both of known but arbitrary (and unrelated) spectrum.

The textbook Wiener filter

Wiener (1939): find a causal $F(\omega)$ such that $\left\langle |\tilde{s}-s|^2 \right\rangle$ is a minimum, with $\tilde{s} = F(\omega)(s+n)$, where s is a signal and n a noise, both of known but arbitrary (and unrelated) spectrum.

Problem: $\langle [(n^* + s^*)F^* - s^*][F(s+n) - s] \rangle = \min.$

Correlations: $N(\tau) = \langle n(t+\tau)n^*(t) \rangle\,;$
$S(\tau) = \langle s(t+\tau)s^*(t) \rangle\,; \quad \langle n(t+\tau)s^*(t) \rangle = 0.$

Answer: $F(\omega)(S(\omega) + N(\omega)) - S(\omega) = 0$

The straightforward solution is $F = S/(S+N)$ but the corresponding $f(t)$ may be noncausal. To make F causal, the **Wiener-Hopf** factorization was invented.

Numerical approaches

to the Wiener-Hopf factorization:

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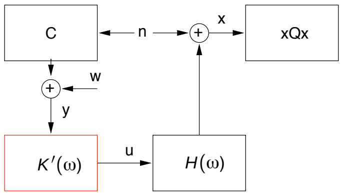
- FFT-based Wiener-Hopf factorization: extremely fast $O(\mathcal{N}_t \log \mathcal{N}_t)$, S.I.S.O.
- Levinson algorithm (most widely used in seismology): fast $O(\mathcal{N}_t^2)$, M.I.M.O.
- standard LU decomposition: slow $O(\mathcal{N}_t^3)$, M.I.M.O., foolproof.

$\mathcal{N}_t$: number of time steps in the discretization.

Equivalent feedforward controller



The feedforward controller K' obeys a quadratic optimization problem; setting its gradient to zero yields a linear system. K is causal if and only if K' is. The closed-loop system is stable if and only if K' is (Youla 1978).

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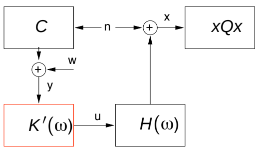
Wiener optimization of a controller



Problem:

$$\langle[(n^*C^*+w^*)K'^*H^*+n^*]Q[n + HK'(Cn+w)] + (n^*C^*+w^*)K'^*RK'(Cn+w)\rangle = \min.$$

Correlations: $N = \langle nn^* \rangle;\quad W = \langle ww^* \rangle;\quad \langle nw^* \rangle = 0$

Answer: $(H^*QH + R)K'(CNC^* + W) + H^*QNC^* = 0$

In frequency space the matrices appearing in this linear system of equations are constant but the resulting K' may be noncausal. Enforcing causality of K' is a **Wiener-Hopf** problem: the relevant SISO or MIMO methodology can be applied.

Advantages of Wiener vs. Kalman optimization

- H can represent a measured input response,
- N can represent a measured correlation, independent of H.
- The state autocorrelation N never appears alone, only as $NC^* = \langle ny^* \rangle$; therefore the algorithm is computationally efficient:
  - convolutions: time $O(\mathcal{N}_s \mathcal{N}_m \log \mathcal{N}_t)$ and $O(\mathcal{N}_s \mathcal{N}_a \log \mathcal{N}_t)$
  - Wiener de-convolution: time $O(\mathcal{N}_m \mathcal{N}_a \mathcal{N}_t^2)$ (with Levinson algorithm)
- rather than Riccati equation for a matrix of size $\mathcal{N}_s \times \mathcal{N}_s$.

$\mathcal{N}_s$: number of states (typically hundreds); $\mathcal{N}_m$: number of measured quantities ($1 \div 3$); $\mathcal{N}_a$: number of actuated quantities ($1 \div 3$); $\mathcal{N}_t$: number of time steps in the discretization (also typically hundreds).

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Dissipation

A physically grounded objective function

The quadratic objective function most frequently adopted as the optimization objective in flow control is the kinetic energy (integral squared velocity). Skin friction of a real turbulent flow is a complicated nonquadratic function and cannot directly be adopted as the optimization objective.
However, there is another quadratic function that gives three distinct advantages:

- Dissipation is a quadratic function that is exactly proportional to skin friction in the mean unperturbed flow.
- Dissipation, in a controlled flow, exactly equals the net energy balance between the work done by skin friction and the work done by the controller, and is thus an objective function directly related to the physical objective.
- Dissipation can be modified, and its modification verified, even in a linear model of turbulence control.

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Motivation

Optimal control of wake instabilities via application of modern control algorithms (Riccati equation) is intractable because of the very large number of degrees of freedom deriving from the discretization of the Navier-Stokes equations.

Motivation

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Optimal control of wake instabilities via application of modern control algorithms (Riccati equation) is intractable because of the very large number of degrees of freedom deriving from the discretization of the Navier-Stokes equations.

An approach based on direct and adjoint eigenvectors makes, at least in the minimal-control-energy problem, mathematically rigorous optimal control a reality.

Application: stabilizing a wake[1]

Screen

[1]see: J. Pralits, T. Bewley & P. Luchini, later in this session.

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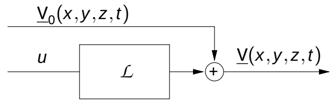
Linear model of a turbulent flow?

$$\underline{V}(x,y,z,t) = \underbrace{\underline{V}_0(x,y,z,t)}_{\text{uncontrolled turbulent flow}} + \underbrace{\mathcal{L}\underbrace{u}_{\text{control input}}}_{\text{small perturbation}}$$



Linear operator $\mathcal{L}$ can represent:

1. Linearized NS problem about the instantaneous flow (diverges in time!)
2. Linearized NS problem about a mean profile (used in past optimal-control approaches)
3. Mean linear response of the turbulent flow (present aim)

Frequency-response representation

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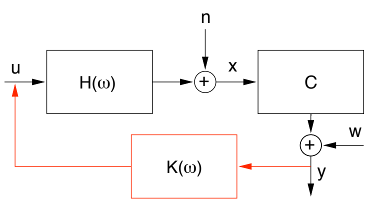

H(ω) is the (numerically) measured mean linear response of the turbulent flow.
y can be any or all wall stress component, **u** any or all wall velocity component.
n is the measured turbulent-flow fluctuation.
Quadratic objective function: $x^*Qx + u^*Ru$ (dissipation)

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The mean linear response[2]

- in frequency domain: $H(\omega)$ is the response to sinusoidal forcing of varying angular frequency ω
- in time domain: $g(t)$ is the impulse response to a Dirac function $\delta(t)$.

Linearity requires the perturbations to be smaller than turbulent fluctuations.
$\Rightarrow$ Conceptual solution: phase-locked averaging (either with impulsive or sinusoidal forcing) to extract deterministic part of the signal out of turbulent noise.

- Main difficulty: how to obtain a sufficiently good signal-to-noise (S/N) ratio.

[2] P. Luchini, M. Quadrio & S. Zuccher, Phys. Fluids **18**, 121702(1–4) (2006)

Our solution: forcing with a random signal

From signal theory: when a **white noise** is passed through a linear filter, the **correlation between input and output** is proportional to the impulse response of the system.

$$R_{oi}(t,x,z) = \int g(t-t', x-x', z-z') R_{ii}(t', x', z')\,\mathrm{d}t'\,\mathrm{d}x'\,\mathrm{d}z'$$

- If $R_{ii} = \delta(t,x,z)$ then $R_{oi} = g(t,x,z)$.
- Turbulent fluctuations will be averaged out just as in phase-locking.
- Forcing power is uniformly distributed (in a statistical sense) over time and space; amplitude required for linearity can be as large as for sinusoidal forcing but the whole response is obtained at once.

Computational parameters

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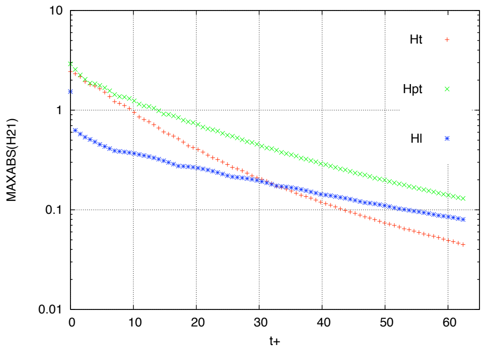
DNS with b.c. for V_i at one wall drawn out of a (space-time) white random distribution with given amplitude.

- Turbulent channel flow at $Re_\tau = 180$
- Standard domain size: $L_x = 4\pi h$, $L_y = 2h$ and $L_z = 4.2h$
- Standard spatial resolution: $N_x = 192$, $N_y = 128$ and $N_z = 128$
- Averaging time $\sim 10-100$ times larger than generally sufficient to collect turbulence statistics.

Comparison between turbulent and linearized-NS response



The complete response function

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The complete correlation function

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A control-kernel example[3]

- Sensor y: **all three stress components** (can be any or all wall stress components). The longitudinal skin-friction component is displayed.
- Actuator u: **wall-normal velocity** (can be any or all wall velocity components).
- Objective function: **dissipation** (can be kinetic energy, weighted kinetic energy, dissipation).
- Control cost $R = 0.1$; measurement noise $W = 0.1$. R and W play a double role as smoothing factors.

[3] as presented at the 2005 APS-DFD Conference

A control-kernel example

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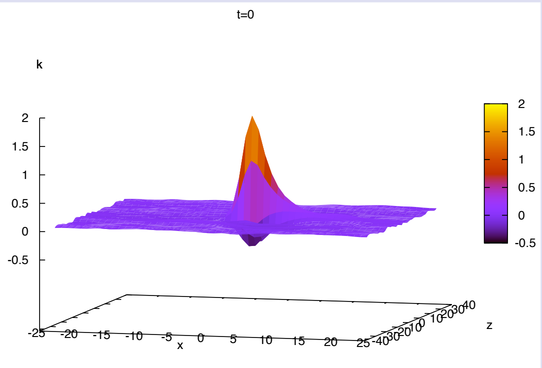
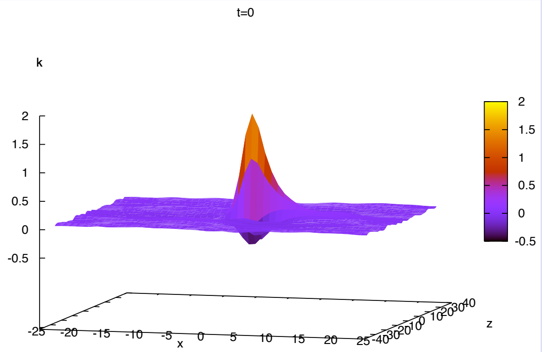
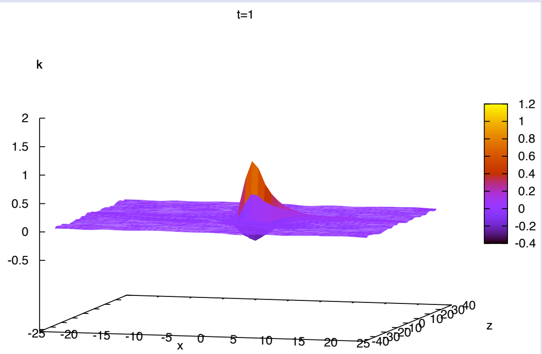


A control-kernel example



A control-kernel example



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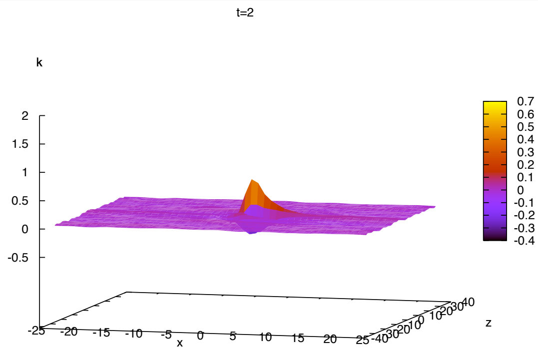

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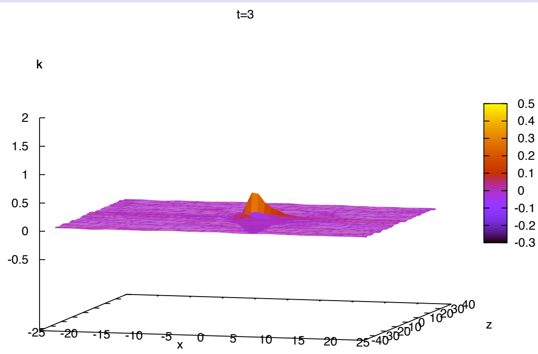

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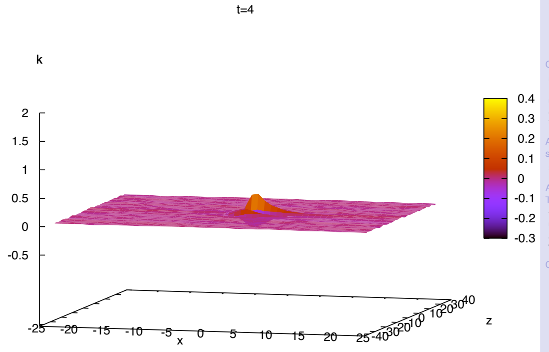

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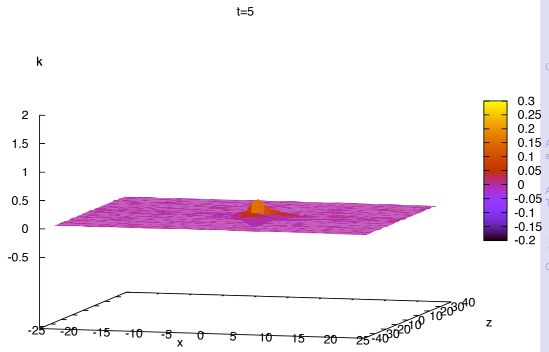

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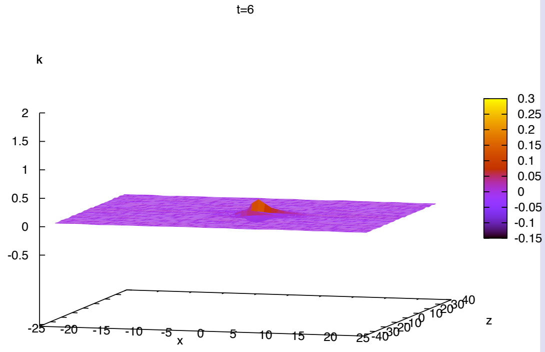

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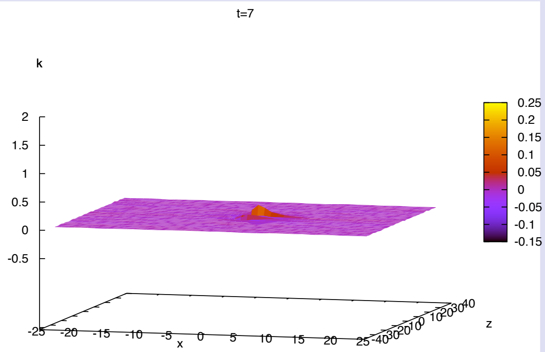

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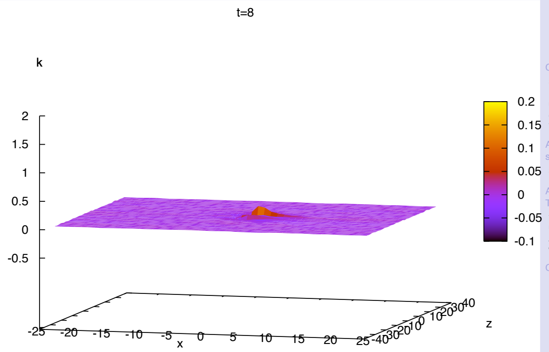

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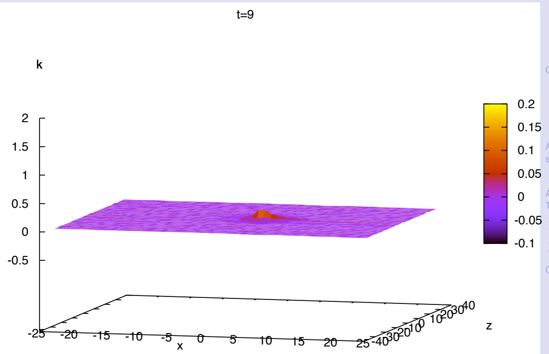

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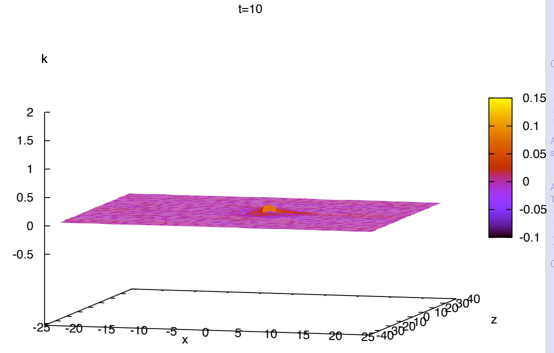

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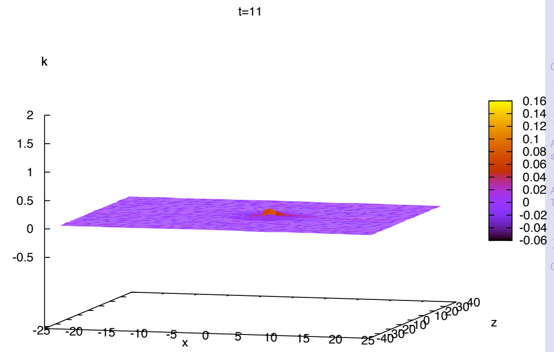

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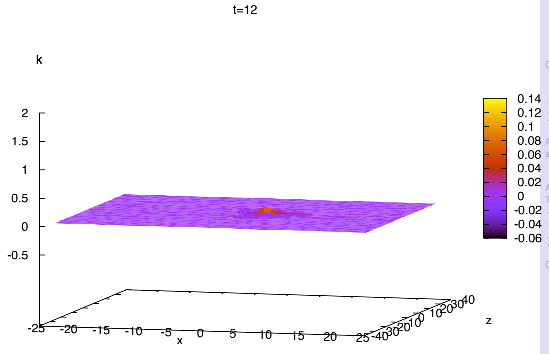

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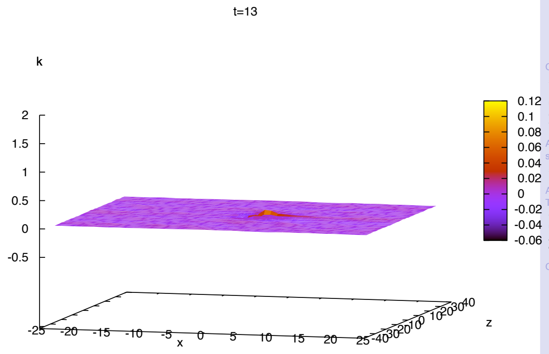

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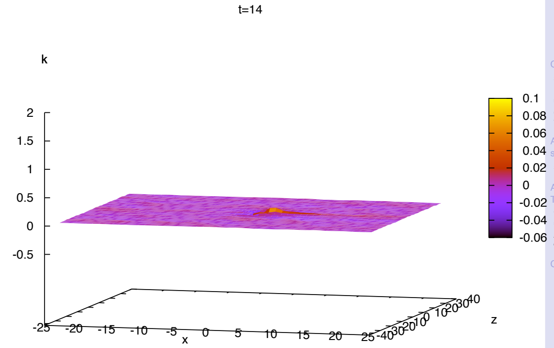

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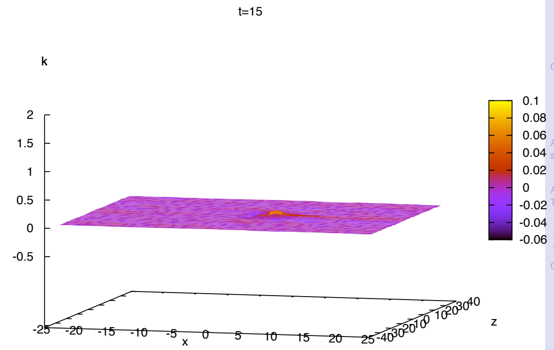

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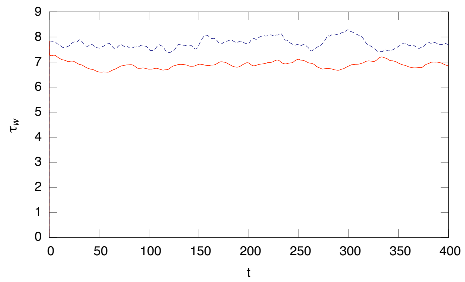
Skin friction vs. time

$4\pi \times 2 \times 4\pi/3, \quad Re_\tau = 180$



Drag reduction: 12 %

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- Flow problems based on continuum mechanics exhibit a large number of degrees of freedom, unaffordable by standardized control algorithms based on the Riccati equation.
- For stability problems, a state formulation based on the direct-adjoint eigenvalue problem opens the way to exact minimal-energy controllers and to approximate solutions based on the leading eigenvectors.
- For turbulent flow, input-output IMC-Wiener optimization provides, in a computationally efficient way, the optimal drag-reducing feedback kernel based on the mean linear response and on actual turbulence statistics.
- Dissipation, a quadratic form, appears to be a more effective objective function than energy for control purposes.