Computer Extended Series Expansion of Dean Flow  
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1. Introduction

In the 1970’s Van Dyke promoted the use of computer extended series expansion solutions within fluid dynamics [4]. Generally, these solutions are valid for low Reynolds number, but sometimes they can be analytically continued to high Reynolds number affording new insight. Though the technique has been successful in many problems, there are a number of cases where its results conflict with other more widely accepted numerical, theoretical and experimental studies. The most cited example is the behaviour of flow through toroidal pipes of finite curvature [5].

In light of the development of new analytic continuation techniques and enhanced computational power, we revisit this problem. Our aim is not only to address a classical problem in fluid dynamics but also to vindicate the use of computer extended series in this field.

2. Dean Flow

We consider a fully developed, steady, incompressible flow driven by a constant pressure gradient in a curved pipe of small curvature. With suitable non-dimensionalisation, the governing equations are the Dean equations:

$$\nabla^4 \psi = \frac{1}{r} \left( \frac{\partial \psi}{\partial \theta} \frac{\partial}{\partial r} - \frac{\partial \psi}{\partial r} \frac{\partial}{\partial \theta} \right) \nabla^2 \psi - \frac{K w}{r} \left( r \cos(\theta) \frac{\partial w}{\partial r} - \sin(\theta) \frac{\partial w}{\partial \theta} \right)$$

$$\nabla^2 w = \frac{1}{r} \left( \frac{\partial \psi}{\partial r} \frac{\partial w}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial w}{\partial r} \right) - 4$$

With no-slip and solid boundary conditions at the pipe $\psi = \frac{\partial \psi}{\partial r} = w = 0$ on $r = 1$. The parameter $K = 2 \left( \frac{W_o a}{\nu} \right)^2 \frac{a}{L}$, is known as the Dean number.

When $K = 0$, the system reduces to Poiseuille flow, with $w = w_0 = 1 - r^2$ and $\psi = \psi_0 = 0$.

From this we construct a perturbation solution:

$$w(r, \theta) = \sum_{n=0}^{\infty} w_n(r, \theta) K^n,$$

$$\psi(r, \theta) = \sum_{n=0}^{\infty} \psi_n(r, \theta) K^n,$$

$$w_n(r, \theta) = \sum_{i=0}^{L_n-1} \sum_{j=0}^{J_n} E_{nij} \begin{cases} \cos [(2i) \theta] r^{2j} & \text{n even} \\ \sin [(2i + 1) \theta] r^{2j+1} & \text{n odd} \end{cases}$$

$$\psi_n(r, \theta) = \sum_{i=0}^{L_n-1} \sum_{j=0}^{J_n-1} C_{nij} \begin{cases} \sin [(2i) \theta] r^{2j} & \text{n even} \\ \cos [(2i + 1) \theta] r^{2j+1} & \text{n odd} \end{cases}$$

The coefficients $E_{n,i,j}$ and $C_{n,i,j}$ can be found exactly to all order.

Here, we consider the flux $Q(K)$:

$$Q(K) = \frac{2}{\pi} \int_0^{2\pi} \int_0^1 w(r, \theta) r dr d\theta = \sum_{n=0}^{\infty} a_n \left( \frac{K}{576} \right)^{2n} \text{ where, } a_n = (576)^{2n} \sum_{j=0}^{J_n} \frac{2 E_{2n0j}}{j + 1}$$

We have calculated up to the $a_{40}$ term as exact rationals, and up to $a_{98}$ as arbitrary precision reals.
The series expansion of $Q(K)$, has a finite radius of convergence. We use analytic continuation techniques to construct a solution which is valid for all $K$.

3.1. Domb-Sykes Analysis and Euler Transformation. We assume the singularity $(K^2_c)$ limiting the radius of convergence is algebraic (e.g. $(K^2 + K^2_c)^\alpha$). It follows that the ratio of consecutive terms must behave linearly in $\frac{1}{n}$:

$$\frac{a_n}{a_{n-1}} = -\frac{1}{K^2_c} \left( 1 - \frac{1 + \alpha}{n} \right)$$

Where, $K^2_c$ is the critical singularity and $\alpha$, the critical exponent. An estimate of $K^2_c$ and $\alpha$ can be found from the reciprocal of the intercept and the terminal gradient at $n = 0$, respectively, of the Domb-Sykes Plot. We find the radius of convergence to be determined by a square-root singularity in the complex plane of $K$. Our estimate of $K_c$ is found to 52 s.f, compared to Van Dyke’s 8. We banish this non-physical singularity by means of an Euler transform (ET):

$$(2) \quad \epsilon = \frac{K^2}{K^2 + K^2_c}$$

The series, re-cast in the parameter $\epsilon$ is analysed in the same manner as the $K$-series. The critical singularity for the ET series is 1, so that a solution has been found which is valid for all $K$. The difficulty in calculating the critical exponent, is explained by a set of confluent singularities at $\epsilon = 1$. Further coefficients are required to distinguish these.

Comparison of the ET to the numerical solution of Siggers [6] (see figure 1), indicates that they are the same. Previously reported differences are a result of the slow convergence of the series.

Numerical studies have found this solution branch to bifurcate [1]. We next study this bifurcation by using the generalised Padé approximants (GPA) method introduced by Drazin and Tourigny [3].

3.2. Generalised Padé Approximants. We assume the series to be algebraic and construct the polynomial system:

$$(3) \quad F_d(K, Q(K)) = \sum_{l=1}^{d} \sum_{m=0}^{l} f_{l-m,m} K^{2(l-m)} Q(K)^m = 0$$

We ensure uniqueness, by truncating the series of $Q(K)$ at $N = \frac{d^2 + 3d - 1}{2}$ and choosing $f_{0,1} = 1$. Solutions $Q(K)$ of (3) are found by standard path continuation techniques.

A system such as (3), can exactly describe algebraic singularities. The number of solutions of the polynomial equation $F_d(K, Q(K))$ increases with $d$. Some of the solutions are spurious, but those that remain must be actual solutions of the Dean Equations.

In Figure (1), we see that the main solution branch for $d = 12$ in (3), agrees with the numerical results of Siggers [6]. This supports the conclusion that series (2) is on the same solution as the numerics.

In Figure (2), we plot the solution branch for $K^2 < 0$. We find that the turning point corresponds exactly with the square-root singularity calculated from the Domb-Sykes analysis. For comparison, we have also altered the numerical scheme of Siggers [6] to allow for complex $K$, $w(r, \theta)$ and $\psi(r, \theta)$. The lower branch in figure (2) is in agreement with numerics.

We have been able to continue this branch around the turning point bifurcation. We find a secondary solution of $Q(K)$ (which corresponds to complex $w(r, \theta)$) which appears to tend to infinity as $K$ tends to zero. Current study are aimed at understanding the asymptotics of this branch and using GPA to construct the multiple solutions of the Dean equations.
4. Conclusion

It is clear from the above work that with appropriate analytic continuation methods, computer extended perturbation solutions are a powerful tool with which to tackle problems in fluid dynamics. The somewhat involved structure at high Reynolds numbers can be approached by solving a succession of relatively simple linear problems. In the case of bifurcation to multiple solutions, the use of the GPA is particularly attractive. Whilst retaining the computational simplicity of working with algebraic systems, it is able to give insight into the bifurcation structure of solutions to the Navier-Stokes equation.

References