Vortex Particle Method for Simulation of Viscous Flows

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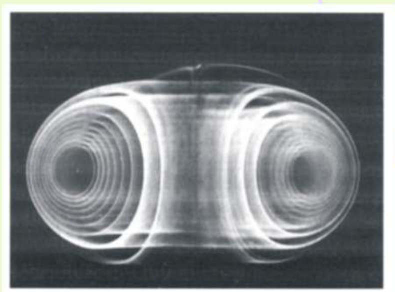
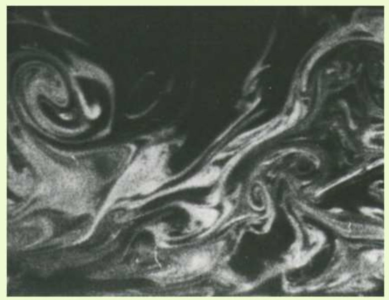
POLAND





Outline of presentation:

- Fundamentals of Vortex Particle Methods - Kinematics of Vorticity,
- Vortex paritcle method (VIC) in 2D
- Eruption of Boudary Layer
- Flying insects
- Water tunel
- Vortex paritcle method (VIC) in 3D –Vortex Rings
- Parallel computations

1. Equations of Motion

- Navier-Stokes equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{u} \qquad (1)$$

$$\nabla \cdot \mathbf{u} = 0$$

Equation (1) can be transformed to the vorticity transport equation:

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = (\omega \cdot \nabla)\mathbf{u} + \nu \Delta \omega$$

where

$$\omega = (\omega_1, \omega_2, \omega_3) = \nabla \times \mathbf{u} = \mathbf{rot}(\mathbf{u})$$

KINEMATCIS OF VORTICITY

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u} = \nabla \times \mathbf{A} \qquad \nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$$

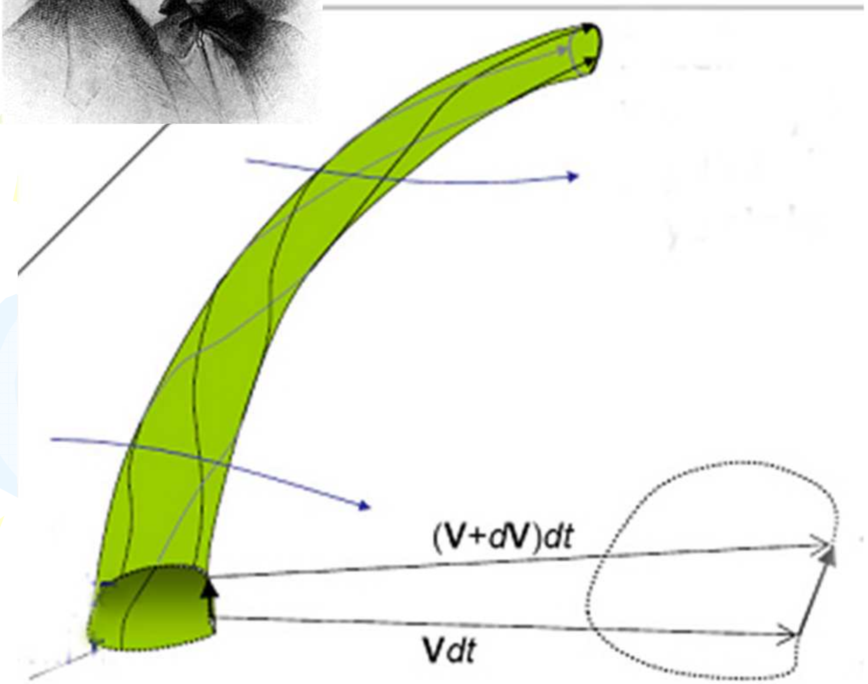
$$\nabla \times \mathbf{u} = \underline{\omega} \qquad \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A}$$

$\mathbf{A} = (A_1, A_2, A_3)$ - vector potential

It is assumend that $: \nabla \cdot \mathbf{A} = 0,$ więc

$$\Delta A_i = -\omega_i, \qquad i = 1, 2, 3$$

Helmholtz theorms



Fluid loop
at time t

The loop at time t+dt
created by the same
fluid particles

1. The strength of vortex tube is uniform along the tube
2. The strength (circulation about any closed circuit C) is invariant in time.
3. Vortex line are material lines

$$\frac{d\,\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x})$$

(1) (3) $\Leftrightarrow$

$$\frac{d\,\underline{\omega}}{dt} = (\underline{\omega}\cdot\nabla)\,\mathbf{v}$$

Two –Dimensional simulation(2D)

$$\mathbf{A} = (0,0,\psi) \qquad \mathbf{u} = \nabla \times \mathbf{A} = (u,v,0) = (\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0)$$

$$\Delta \psi = -\omega, \quad \text{(Poisson equation)} \qquad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega, \text{ Helmholtz equations.}$$

$$\text{If } \nu = 0, \text{ (invicid fluid) then: } \partial_t \omega + u \cdot \nabla \omega = \frac{\mathbf{d}\,\omega}{\mathbf{d}\,t} = 0,$$

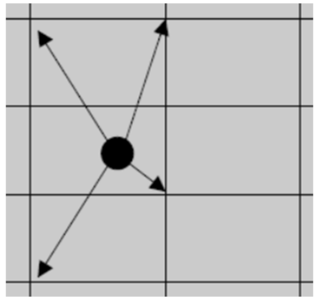
So in 2D the vorticity is constant along the trajectory

Computationl algorithm in 2D:

1. Particle approximation and redistibution to the grid nodes :

$$\omega(\mathbf{x}) = \sum_{p=1}^{N} \alpha_p \, \delta(\mathbf{x} - \mathbf{x}_p); \quad \alpha_{p(i)} = \int_{V_p} \omega_i(\mathbf{x}) d\mathbf{x} \approx h^2 \omega_i(\mathbf{x}_p)$$

$$\omega_j = \sum_p \alpha_p \Lambda_j(\mathbf{x}_p)$$



2. Solution of the Poisson equation for stream function and calculation of the velocity

$$\Delta\psi = -\omega, \qquad u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

3. Displacements of the particles :

$$\frac{d\,\mathbf{x}}{dt} = \mathbf{u}$$

4. Simulation of viscosity and realisation of non-slip boundary condition

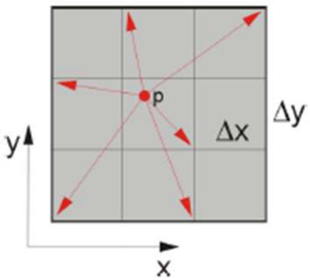
Redistribution

Interpolation functions

After particles movement, in order to solve the diffusion equation, it is necessary to transform the information about vorticity from the particles onto mesh nodes

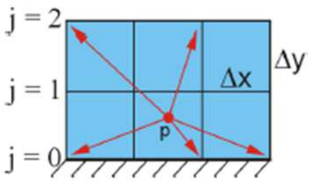
$$\omega_j = \frac{1}{h^2} \sum_p \Gamma_p \varphi(\frac{x - x_p}{h})$$

- Particle inside of the domain



$$\varphi(x) = \begin{cases} 1 - \frac{5}{2}x^2 + \frac{3}{2}|x|^3 & |x| < 1 \\ \frac{1}{2}(2 - |x|)^2(1 - |x|) & 1 \leqslant |x| \leqslant 2 \\ 0 & |x| > 2 \end{cases}$$

- Particle near the wall



$$\varphi(x) = \begin{cases} 1 - \frac{1}{2}x^2 - \frac{3}{2}|x| & j = 0, |x| \leqslant 1 \\ -x^2 + 2|x| & j = 1, |x| \leqslant 1 \\ \frac{1}{2}x^2 - \frac{1}{2}|x| & j = 2, |x| \leqslant 1 \end{cases}$$

Both interpolation kernels conserve three first moments $I_\alpha = \sum_p x^\alpha \Gamma_p, \quad \alpha = 1, 2, 3$

Simulation of viscosity

1. Stochastic approach (Chorin, JFM 1973):

The particle path is regarded as a stochastic process define by Ito stochastic differential eqution:

$$dX(\mathbf{x}_p, t) = \boldsymbol{u}(\mathbf{x}_p, t)dt + \sqrt{2\nu}\, d\boldsymbol{W}(\mathbf{x}_p, t), \quad p = 1, \dots N$$

$$\boldsymbol{x}_p^{n+1/2} = \boldsymbol{x}_p^n + \Delta t \boldsymbol{u}^n(\boldsymbol{x}_p)$$

$$\boldsymbol{x}_p^{n+1} = \boldsymbol{x}_p^{n+1/2} + \sqrt{2\nu\Delta t}\,\boldsymbol{N}_p$$

$$\mathrm{N}^{(1)} = \mathbf{cos}(2\pi \mathrm{U}^{(1)})\sqrt{(-2\,\mathbf{ln}(\mathrm{U}^{(2)})}$$

$$\mathrm{N}^{(2)} = \mathbf{sin}(2\pi \mathrm{U}^{(1)})\sqrt{(-2\,\mathbf{ln}(\mathrm{U}^{(2)})}$$

2. Simulation of the viscosity by Particle-Strength Exchange (PSE) method

$$\nu \Delta_h \omega\Big|_p = \nu \frac{1}{h^2} \sum_{q=1}^{N} (\alpha_q - \alpha_p) \Lambda\left(\frac{\mathbf{x}_p - \mathbf{x}_p}{h}\right)$$

Realisation of the no-slip boundary condition

No-slip condition is relized by generation of the proper amount of the vorticity on the wall. The distribution of the vorticity inside of the flow domain generates non-zero tangent velocity at the wall u_s.

This undesirable tangent velocity can be cancel by proper vorticity flux:

$$\frac{\partial \omega}{\partial t} = \nu \Delta \omega,$$

$$\omega(x, y, 0) = 0,$$

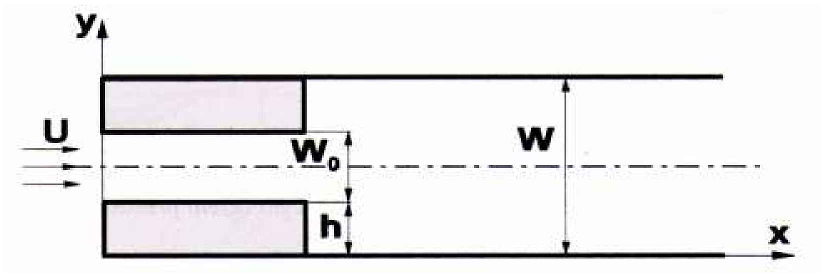
$$\frac{\partial \omega}{\partial n} = -\frac{\gamma}{\delta t \cdot \nu}.$$

$$\left.\frac{du}{dt}\right|_{Wall} \approx \frac{u(t+\Delta t) - u(t)}{\Delta t} = -\frac{\partial p}{\partial x} - \nu \frac{\partial \omega}{\partial y}$$

$$\frac{\partial \omega}{\partial y} = -\frac{u_s}{\nu \Delta t}$$

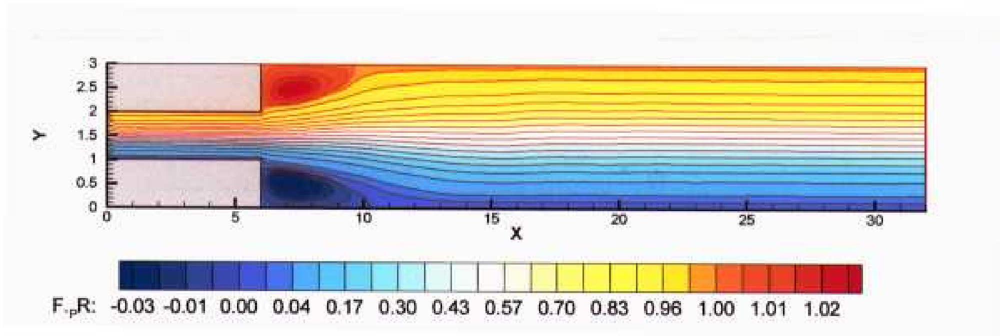
$$\omega_{new} = \omega_{old} + \omega_{gen}$$

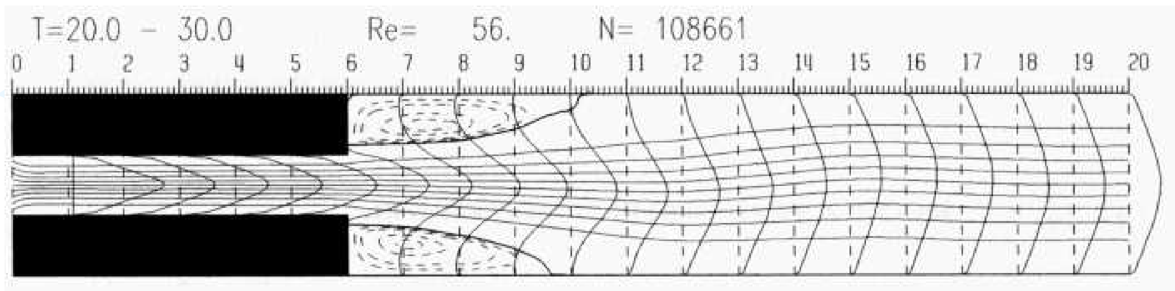
Flow in channel with symmetric sudden expansion



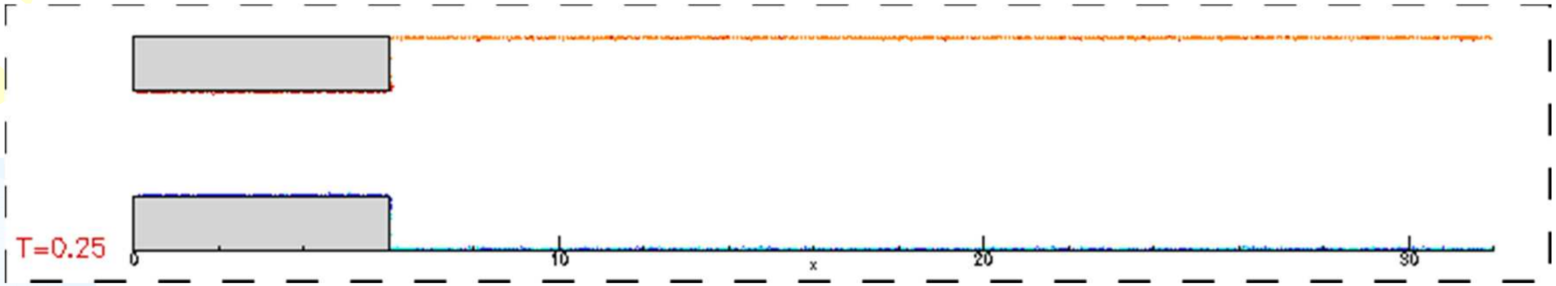
$$\mathbf{Re} = \frac{Uh}{\nu}$$

Re=56





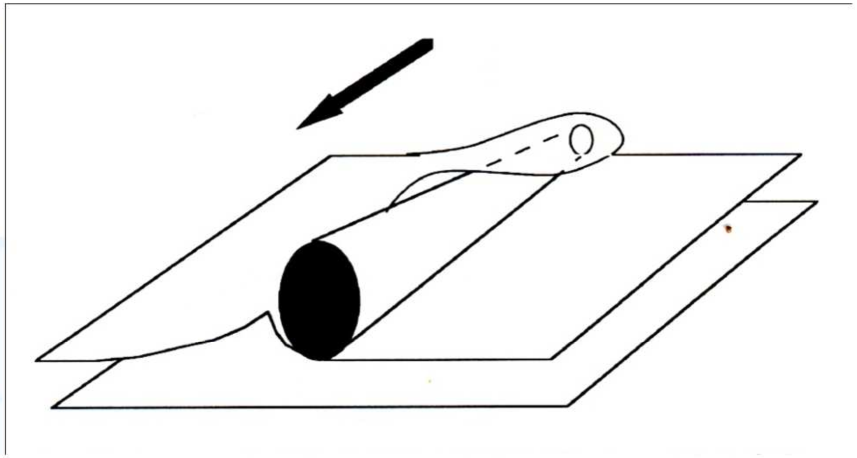
Flow in channel with symmetric sudden expansion cont.

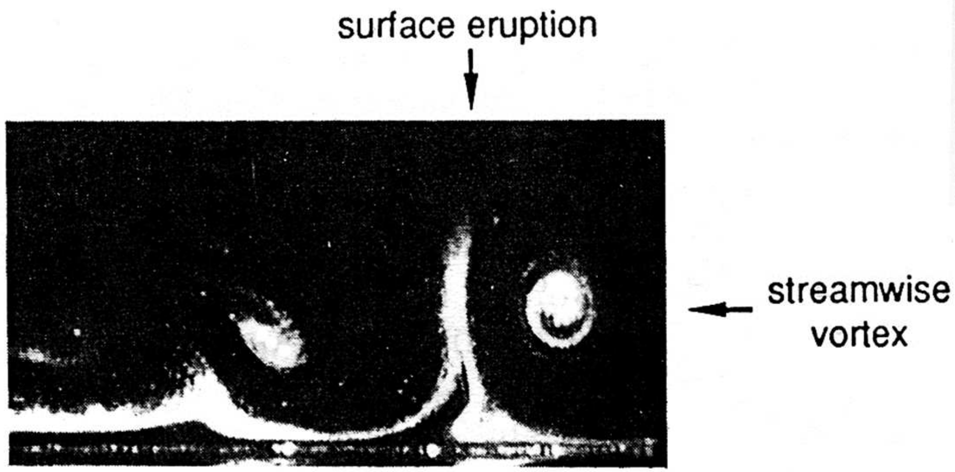


H.Kudela, Task Quart,3 1999

(Experiment by Durst,Melling, Whitelaw, JFM 1974)

Eruption of the Boundary Layer-Motivation





Doligalski, Smith, Walker, 1994, Ann. Rev. Fluid Mech.

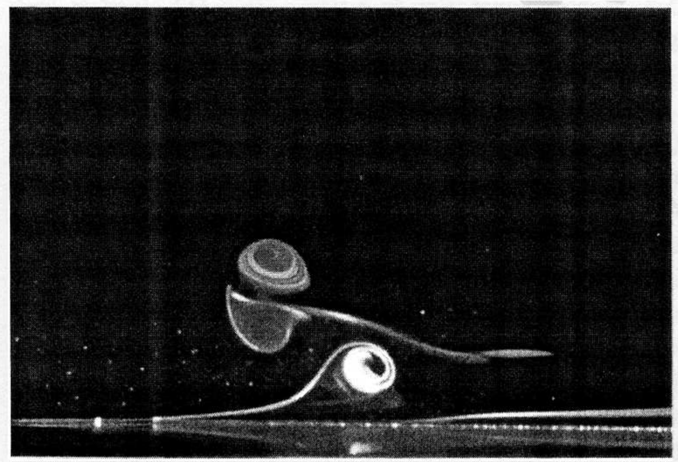
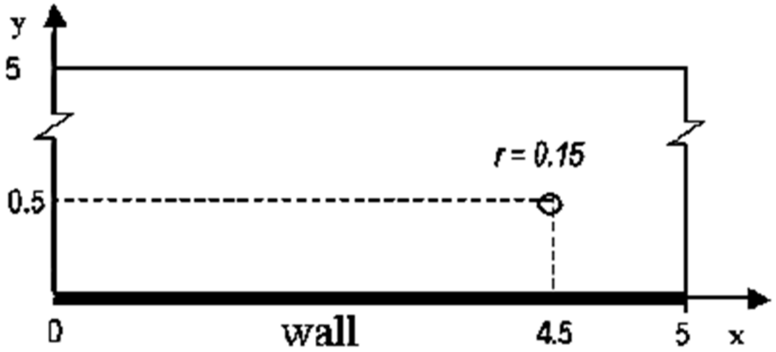


Fig. 20. Experiment demonstrating secondary vortex birth. Photo courtesy of M. Stanislas and P. DuPont, Ecole Centrale de Lille. Panton, 2001, JPAS

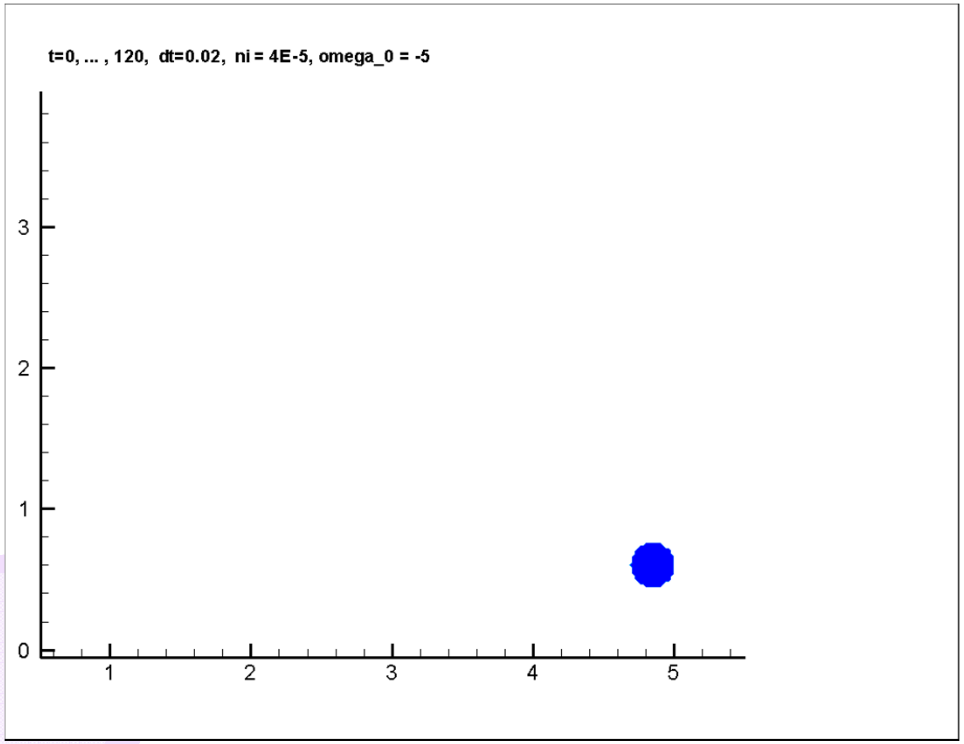
Problem formulation

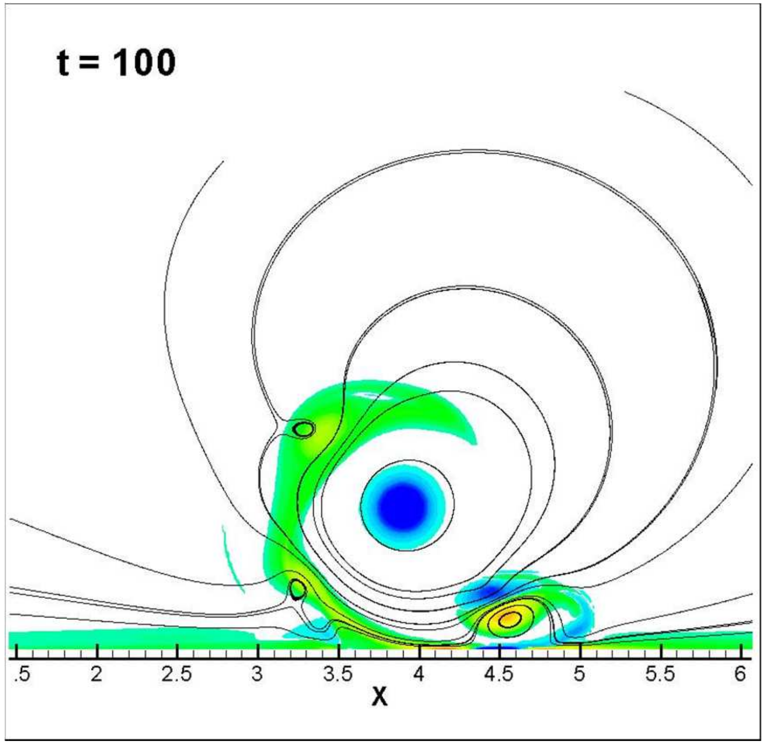


$$\omega_t + \mathbf{u}\cdot\nabla\omega = \nu\Delta\omega$$

$$\Delta\psi = -\omega, \quad u = \partial_y\psi, \quad v = -\partial_x\psi \qquad (1)$$

$$\psi|_{x=0} = \psi|_{x=L}, \qquad \psi|_{y=0} = \psi|_{y=top} = 0$$

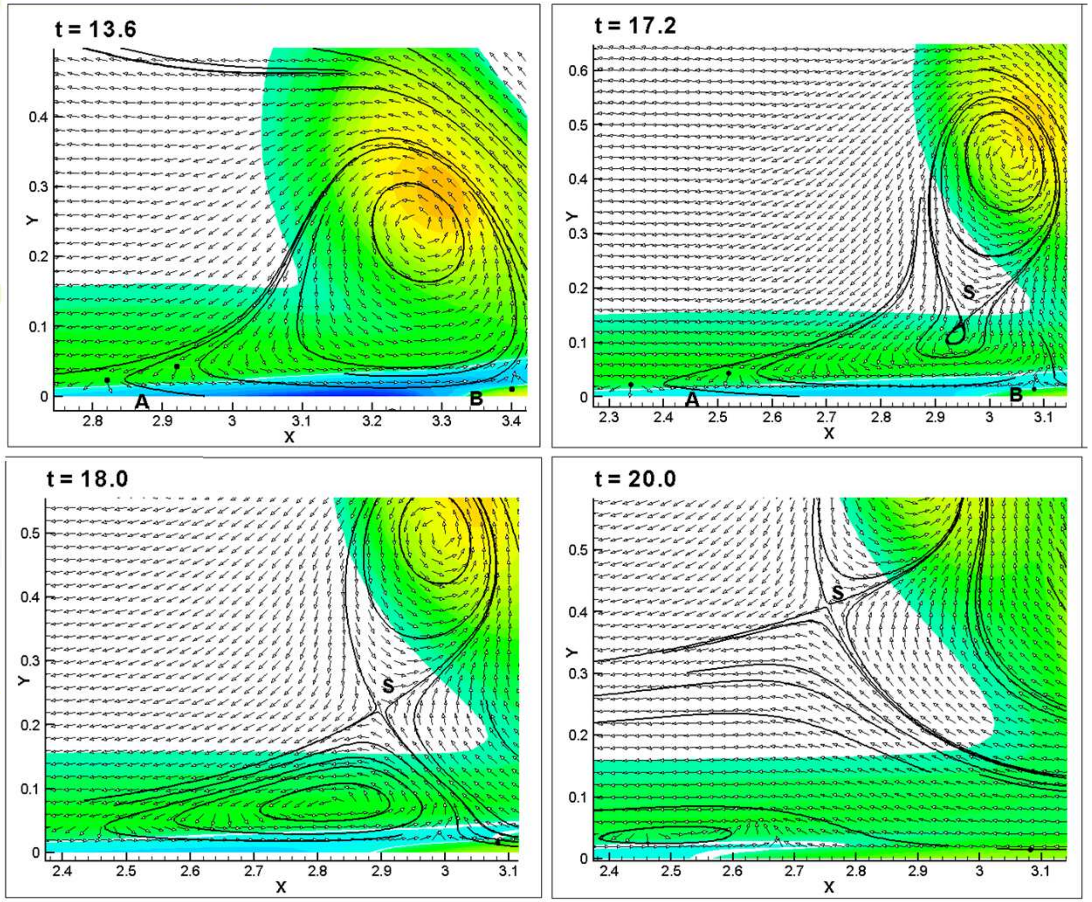




H.Kudela, Z. Malecha
Fluid Dyn. Res., 41,2009

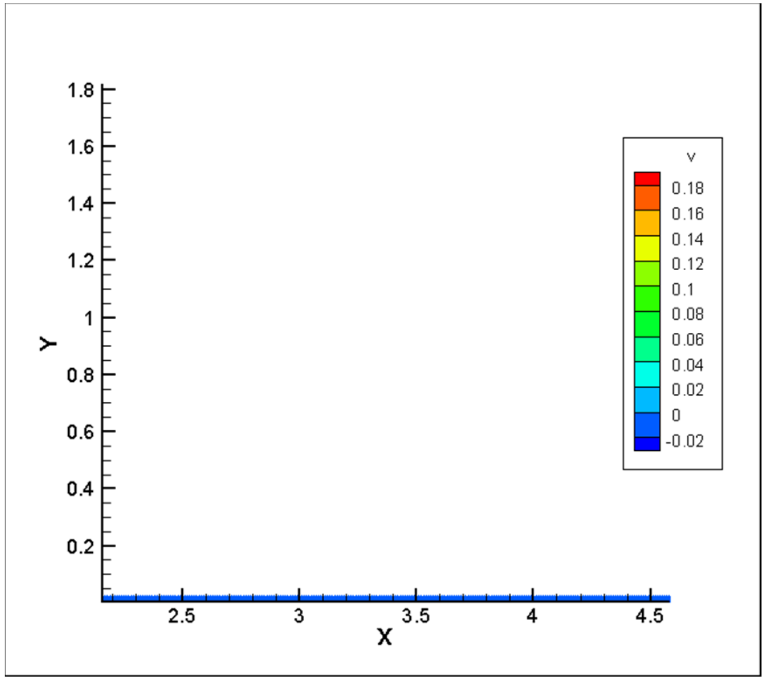
Re = 17670; ν = 0.0002 $Re=\Gamma/\nu$

The sequence of induced secondary vortex structures

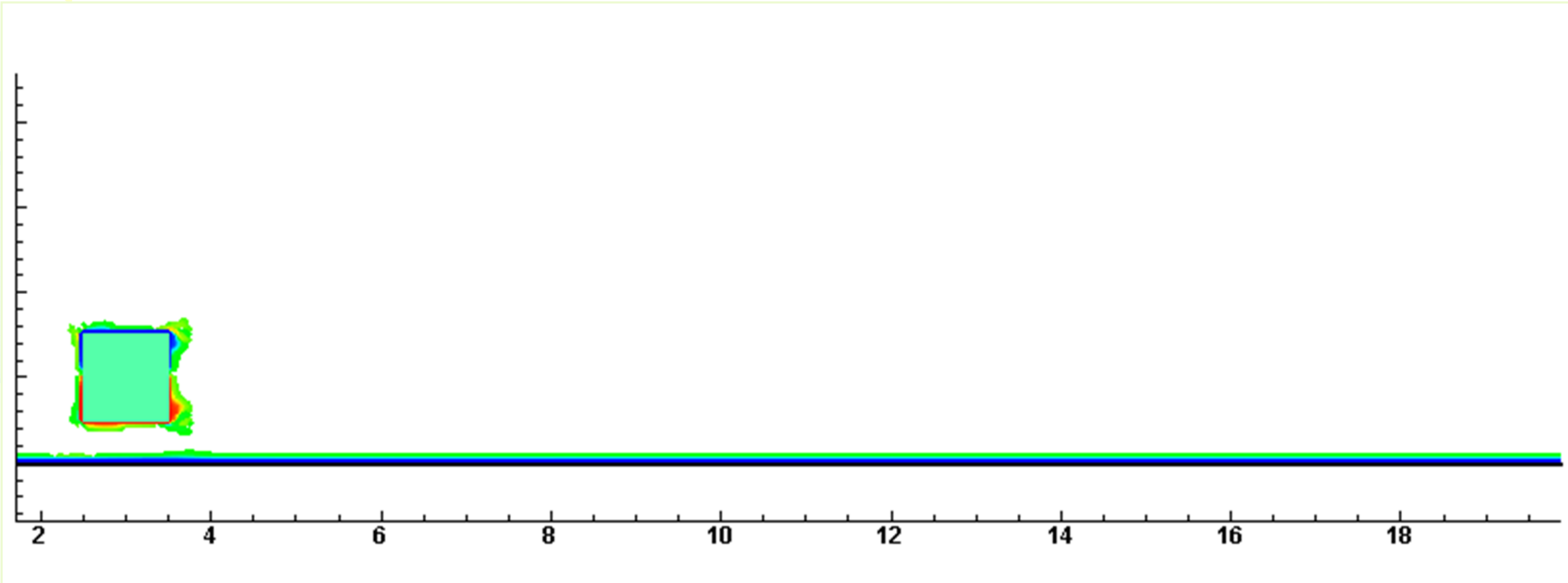


Streamlines with the velocity directions
and vorticity in the fond.

Animation of passive markers
from the boundary region



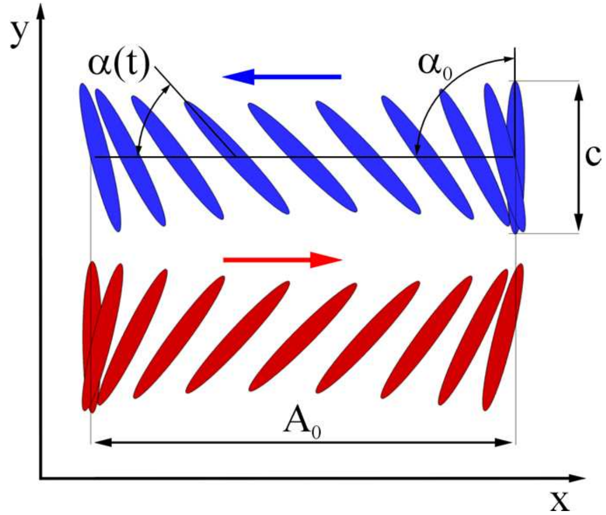
Interaction of the Vortex with the Wall



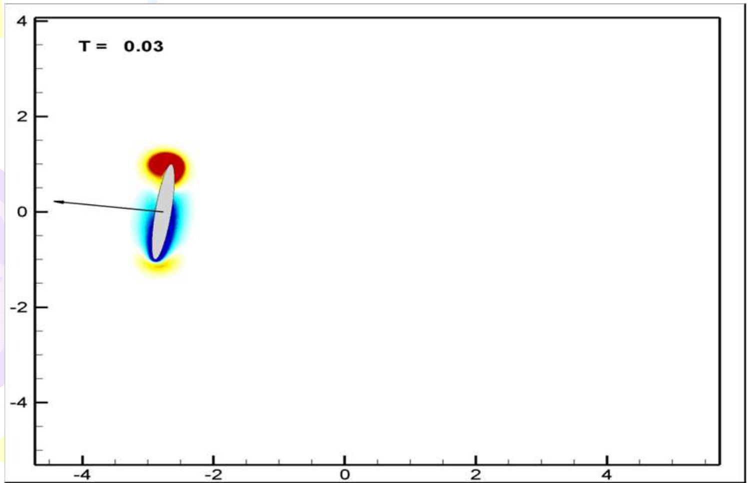
Z. Malecha, *PhD. Thesis, 2009*

Flying insects

2D approximation



Re=75 vorticity field



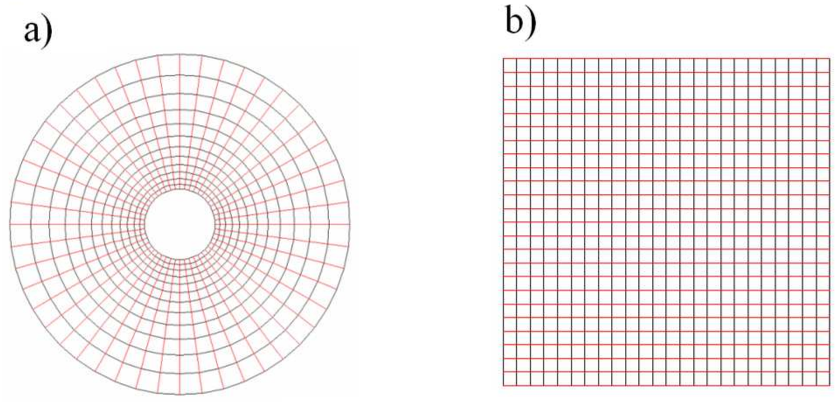
Center of the wing movement

$$\alpha(t) = \alpha_0 + \alpha_A \cos(2\pi f t)$$

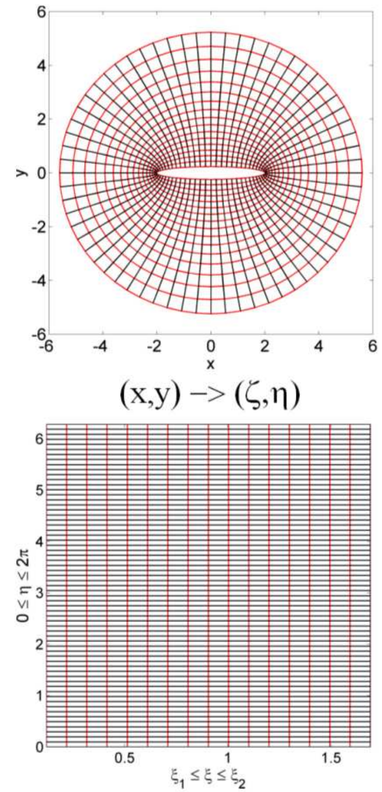
$$[x(t), y(t)] = A_0 \cos(2\pi f t)[\cos(\beta), \sin(\beta)]$$

T. Kozlowski, *PhD. Thesis, 2011*

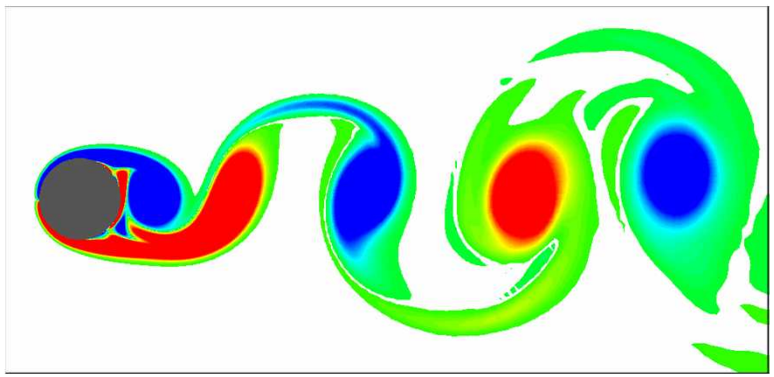
Conformal Mapping

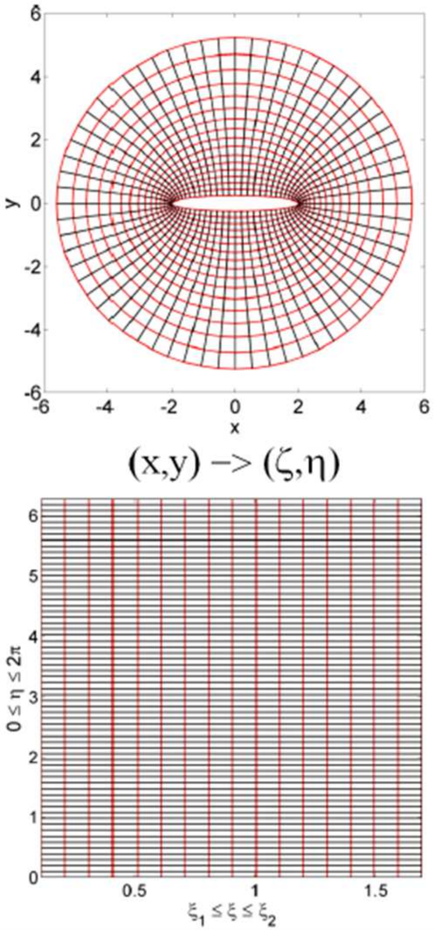


(x,y) => (ζ,η)



$$x + iy = \exp(r + i\theta)$$





In new variables (ξ, η) the equations of motion are

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial \xi} + v\frac{\partial \omega}{\partial \eta} = \frac{\nu}{J}\Delta\omega$$
$$\Delta\psi = -J\omega$$

where J denotes Jacobian of the conformal transformation

$$J = \det\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}$$

The velocity field is determined with the formulas

$$u = \frac{1}{J}\frac{\partial \psi}{\partial \eta}, \quad v = -\frac{1}{J}\frac{\partial \psi}{\partial \xi}$$

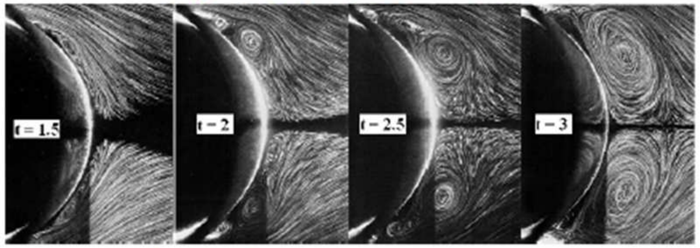
The main advantage of conformal mapping application:

by the conformal mapping the physical non-rectangular flow region (x, y) is replaced by the rectangular one (ξ, η), in which the fast Poisson solvers can be used.

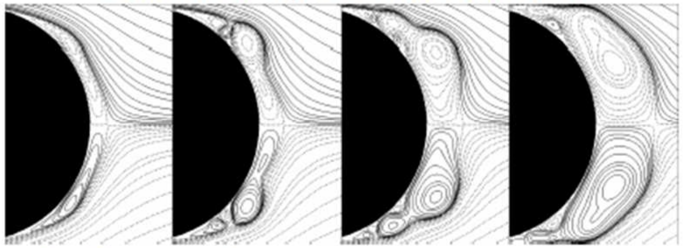
Test of the VIC method

- **Flow over cylinder**

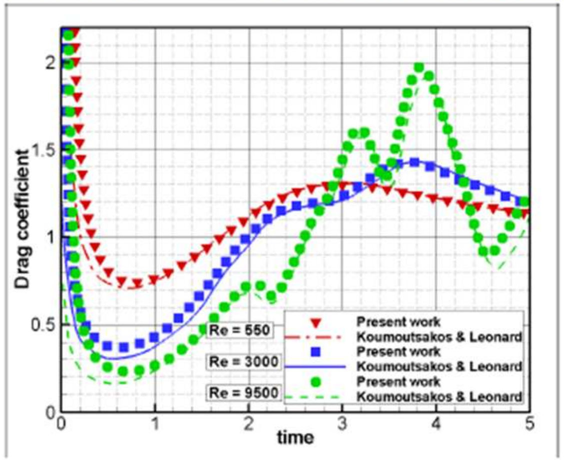
experiment, $Re = 9500$



calculation, stream function

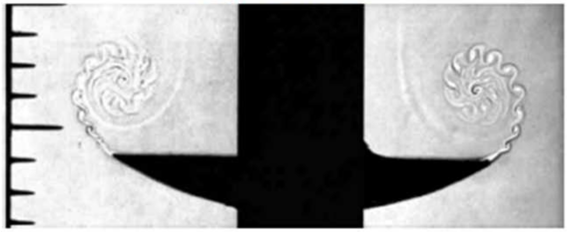


calculation, drag coefficient comparison

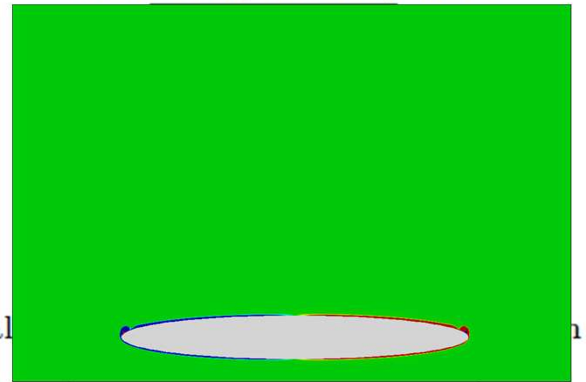


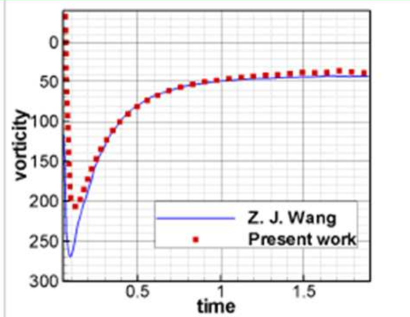
- **Flow over ellipse, $Re = 10000$**

experiment



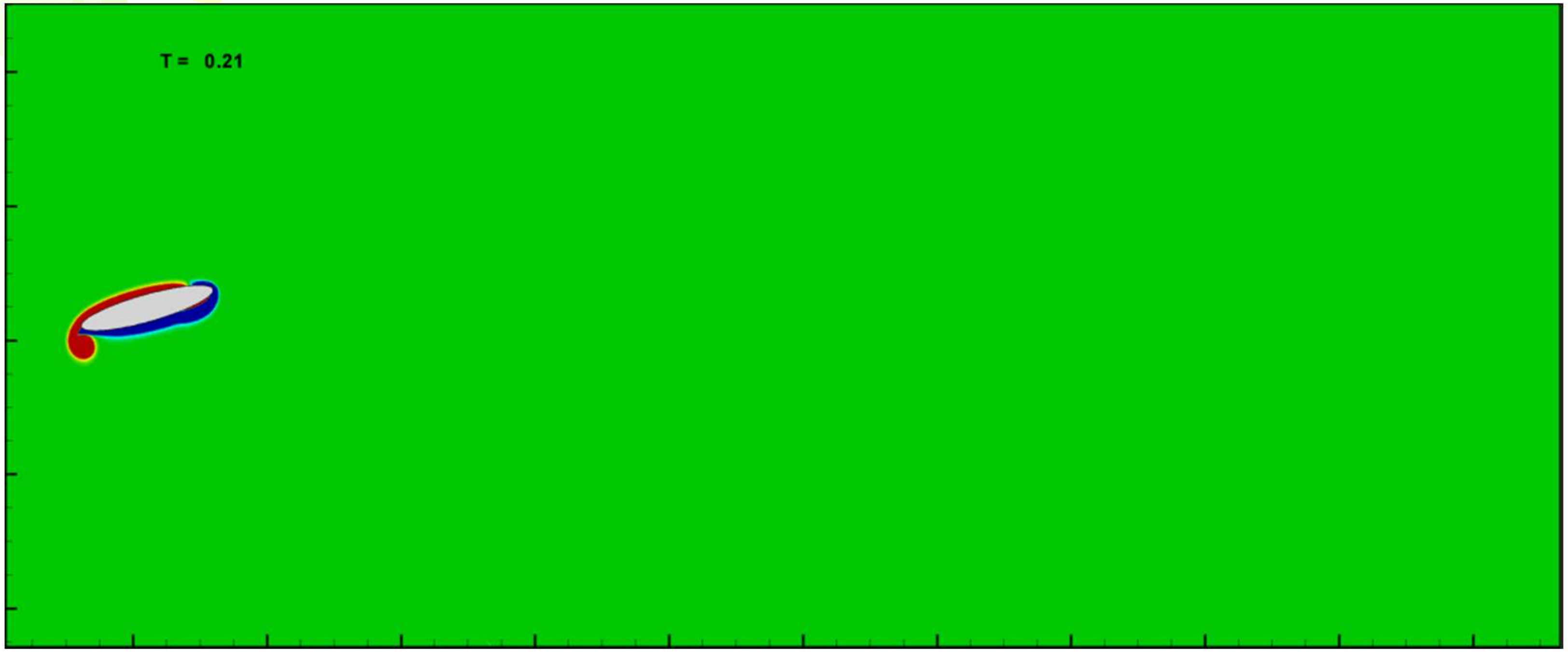
calculation, vorticity field



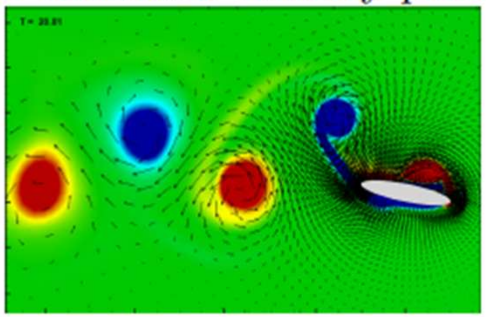


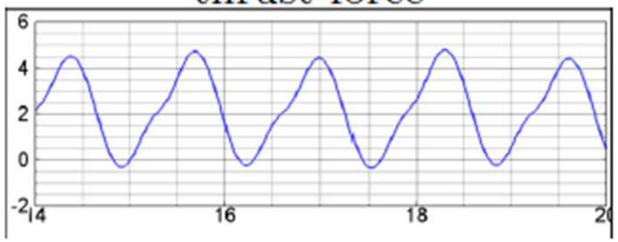
Kudela, Kozłowski, *J. Theor. Appl. Mech.,47, 2009*

The change of topolgy of vortex street

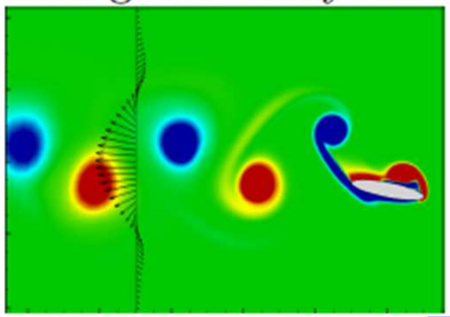


the transient velocity profile





averaged velocity field

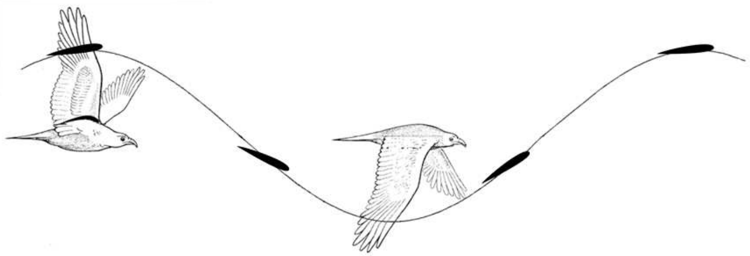


AIAA-paper, 2009

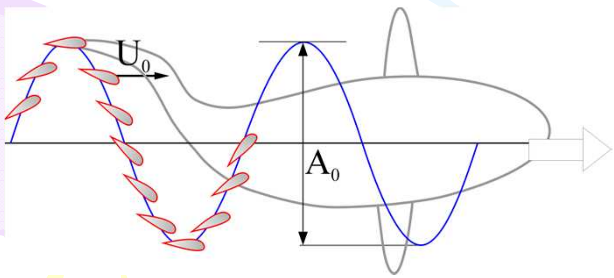
Flapping scheme in living nature

Insects

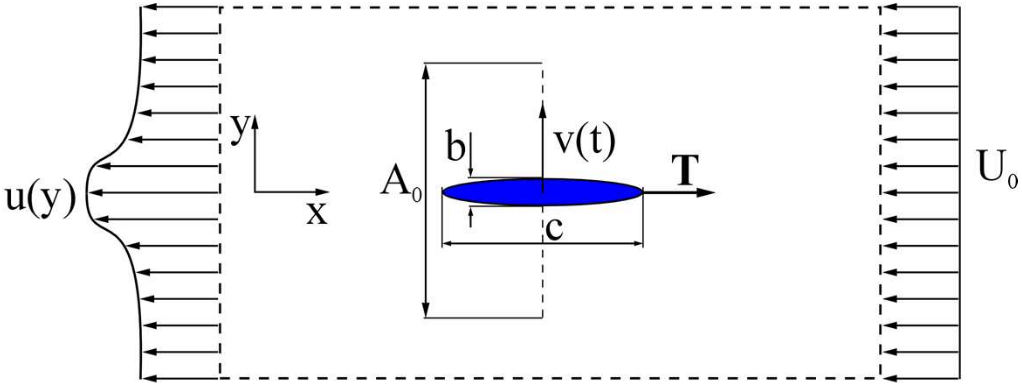
Birds



Fishes



Simplification



Foil oscillation

$$y(t) = \frac{A_0}{2}\sin(2\pi f t)$$

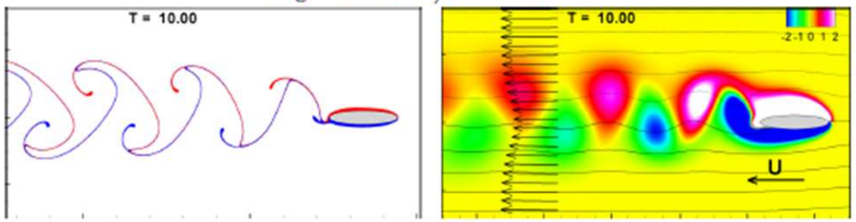
Flapping system can be analysed in terms of three main non-dimensional parameters

$$Re = \frac{U_0 c}{\nu}, \quad St = \frac{fc}{U_0}, \quad A_c = \frac{A_0}{c}$$

Transitions of the vortex street of a flapping foil, *Re* = 100

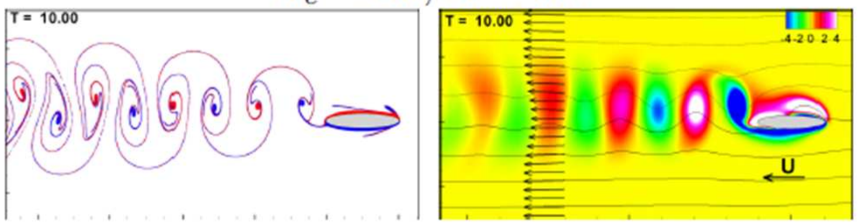
❶ Karman vortex street

$A_c = 0.25, \; St = 0.8$



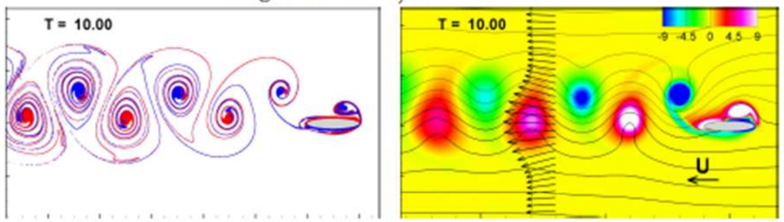
❷ aligned vortices

$A_c = 0.5, \; St = 0.8$



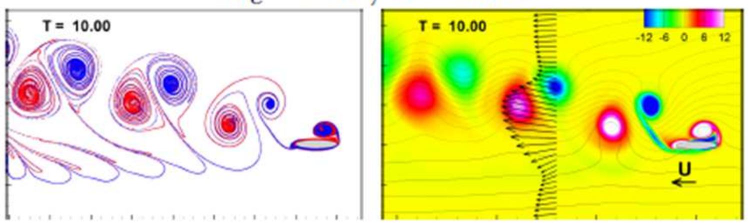
❸ reversed Karman vortex street

$A_c = 0.75, \; St = 0.8$



❹ deflected reversed Karman vortex street

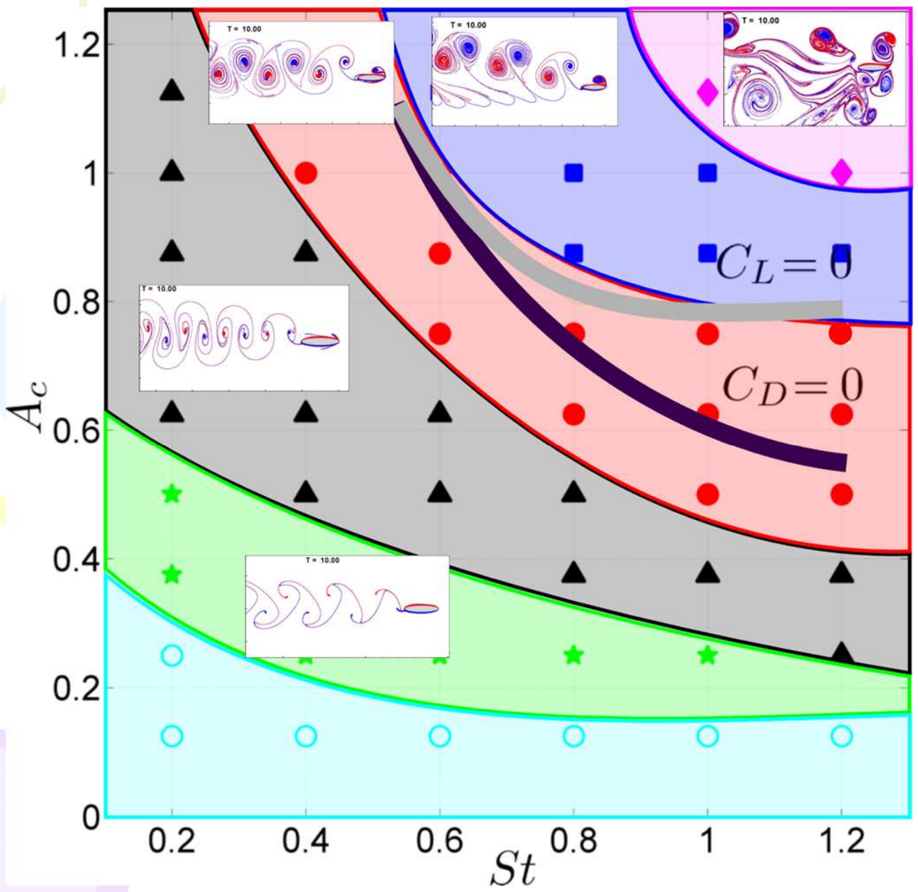
$A_c = 1.0, \; St = 0.8$



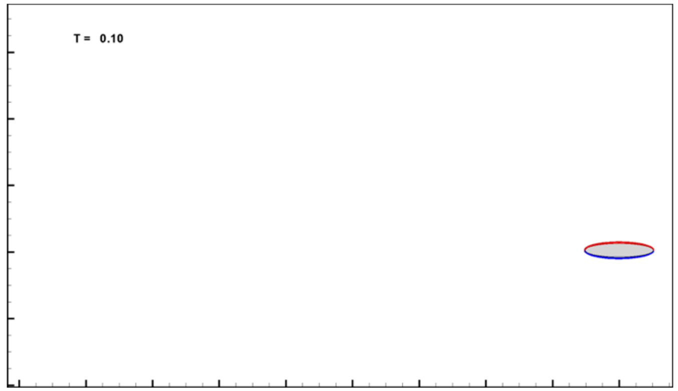
$$St = \frac{fc}{U_0}, \qquad A_c = \frac{A_0}{c}$$

Kudela, Kozlowski, *Chem. And Process Eng. ,31,2010*

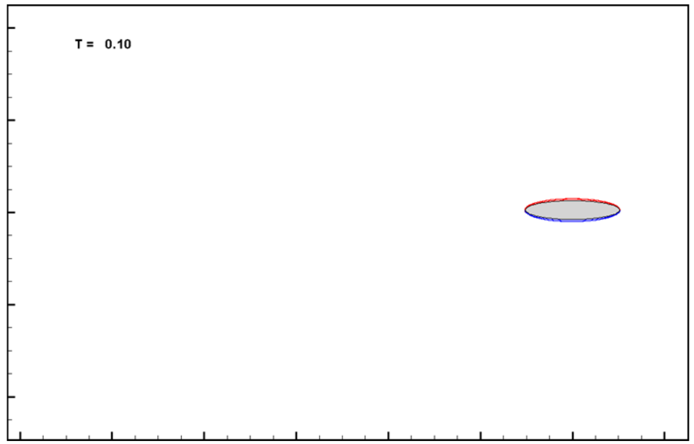
Phase transition diagram



Deflected vortex wake, thrust and lift force



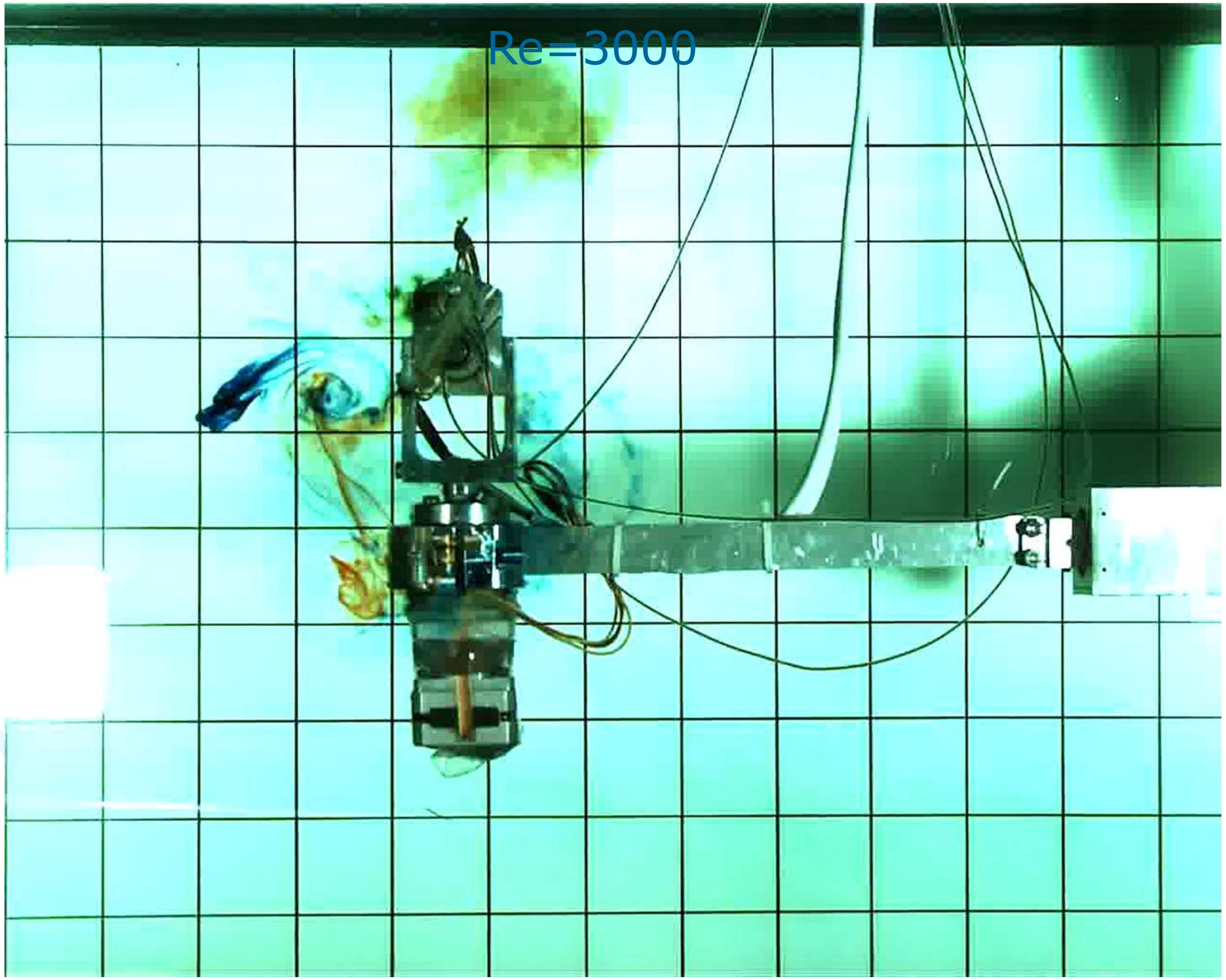
Chaotic vortex wake



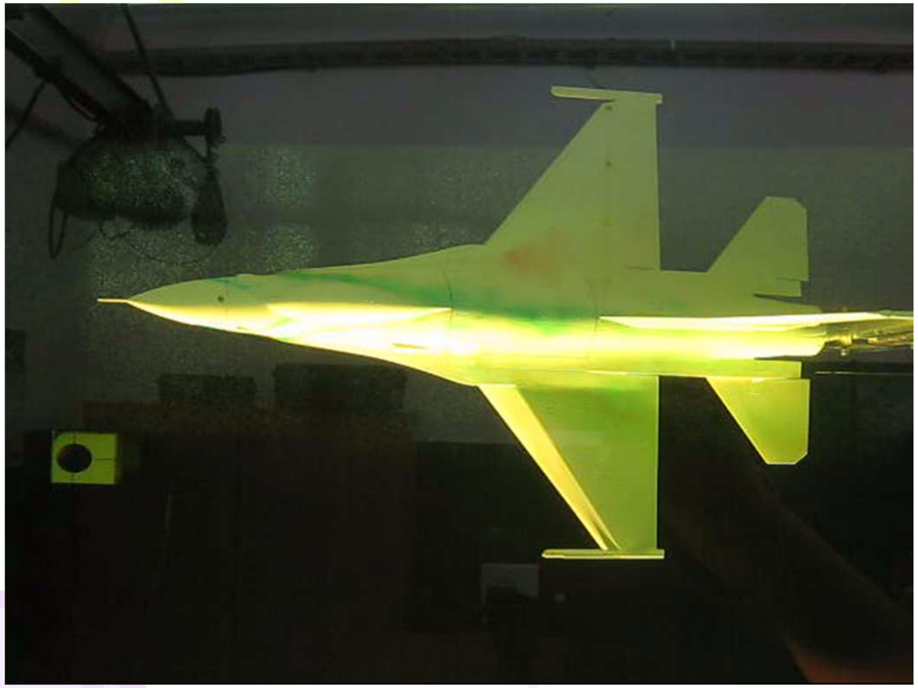
Departament of Aerospace Engineering
prof. K. Sibilski group

The Rolling Hills Research Corporation Flow Visualization Water Tunnel

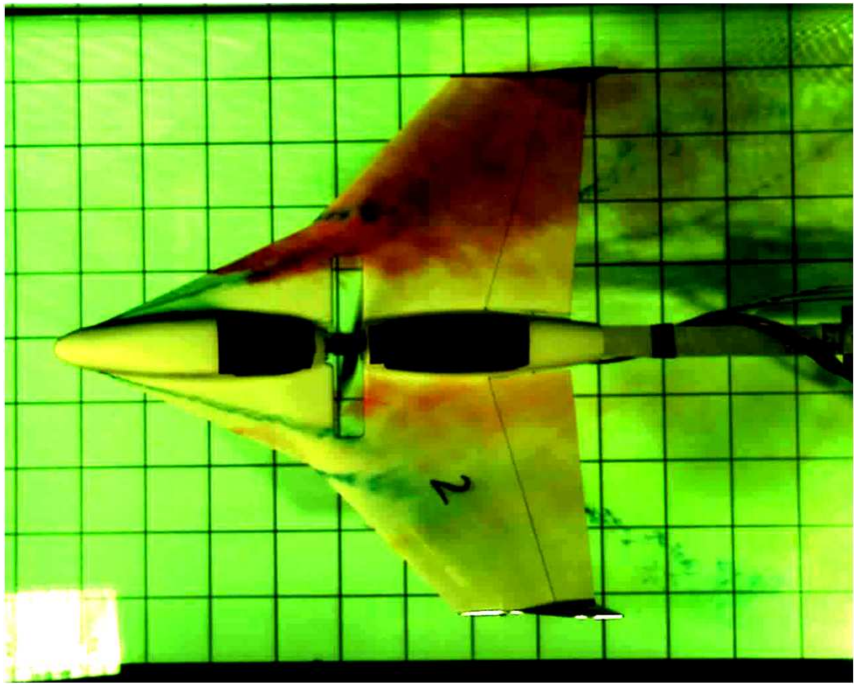
Flapping mechanism



Flow Visualizations



F-16



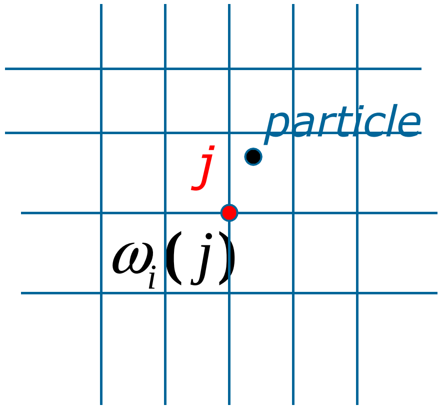
UAV project

Department of Aerospace Enginnering, prof. K. Sibilski, dr Gronczewski

3D VORTEX IN-CELL

The velocity of the particles are caclulated by solving Poisson equations by finite diffrence method for vector potential:

$$\Delta A_i = -\omega_i, \quad i = 1, 2, 3$$



To obtain the grid values for $\omega_i(j)$ the redistribution of weights particles process must by done:

$$\omega_{i,j} = \begin{cases} \dfrac{\sum_p \alpha_p \varphi_j(\mathbf{x}_p)}{J_j}, & J_j \neq 0 \\ 0, & J_j = 0 \end{cases}$$

$$J_j = \sum_p h^3 \varphi_j(\mathbf{x}_p) \approx h^3$$

φ_j – 3D B-spline of 3th order

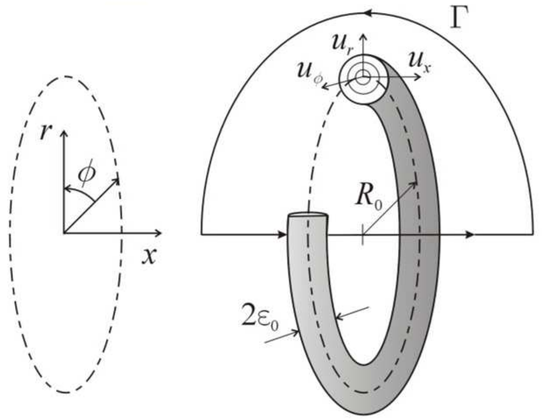
Velocity on the grid nodes is calcualated by finite differences

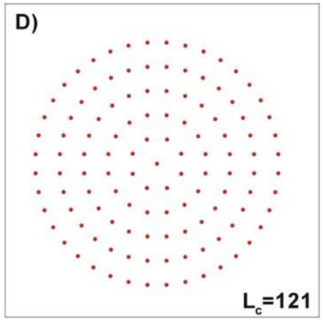
$$\mathbf{u}(\mathbf{x_j}) = \nabla \times \mathbf{A}$$

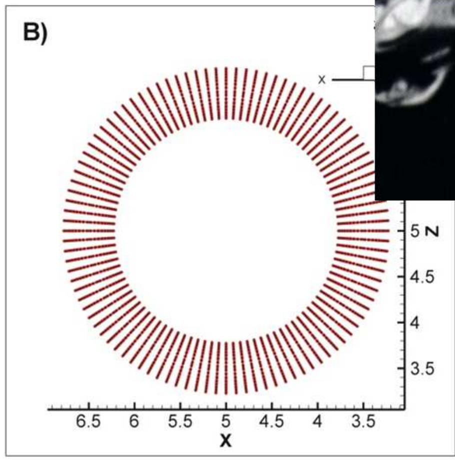
Velocity of the particles is obtained by the interpolation:

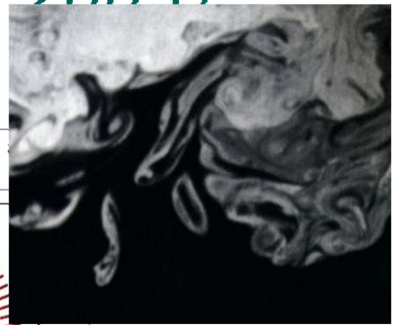
$$\mathbf{u}(\mathbf{x_p}) = \sum_j \mathbf{u}(\mathbf{x_j}) L_j(\mathbf{x_p})$$

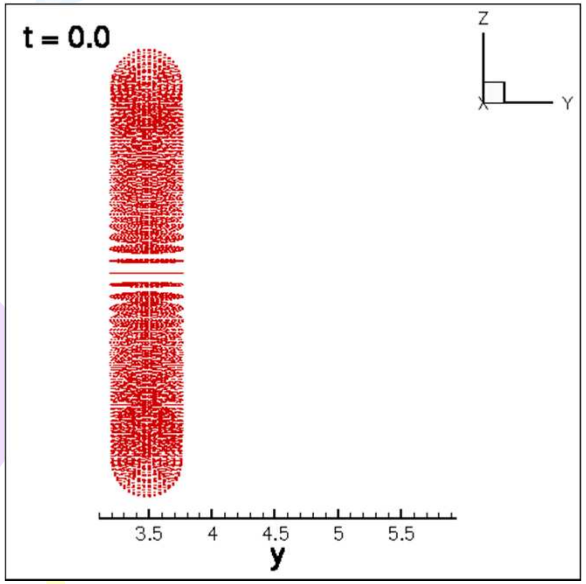
VORTEX RING MOTION (Regucki, Ph.D 2003)











Geometrical parameters of the ring:

- inner radius $r_0 = 0.3$
- outer radius $R_0 = 1.5$
- circulation $\Gamma = 1.0$

-number of the grid nodes: i = j = k = 101

- $\Delta x = \Delta y = \Delta z = h = 0.1$
- time step: $\Delta t = 0.02$
- number of vorticity-particles: N = 12100

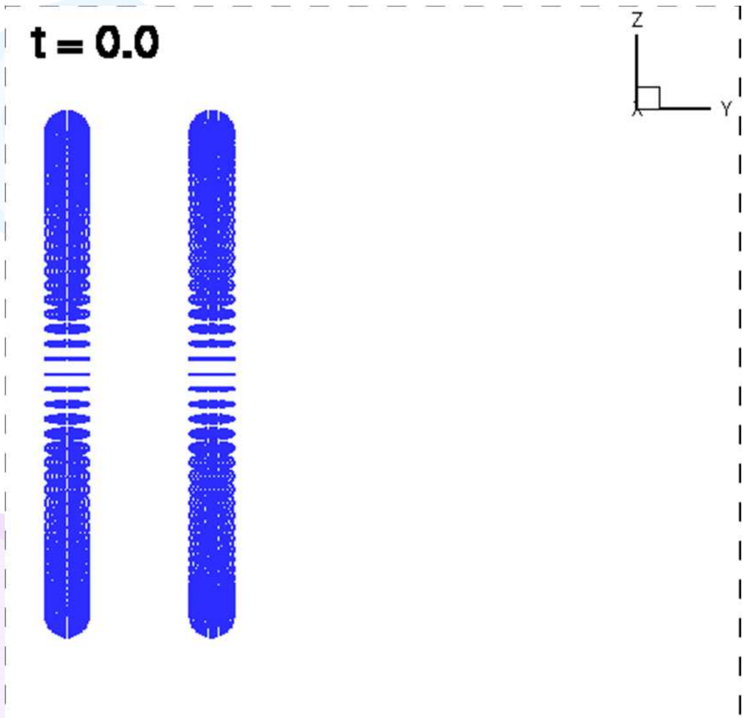
VORTEX GAME **(Leap-frogging)**

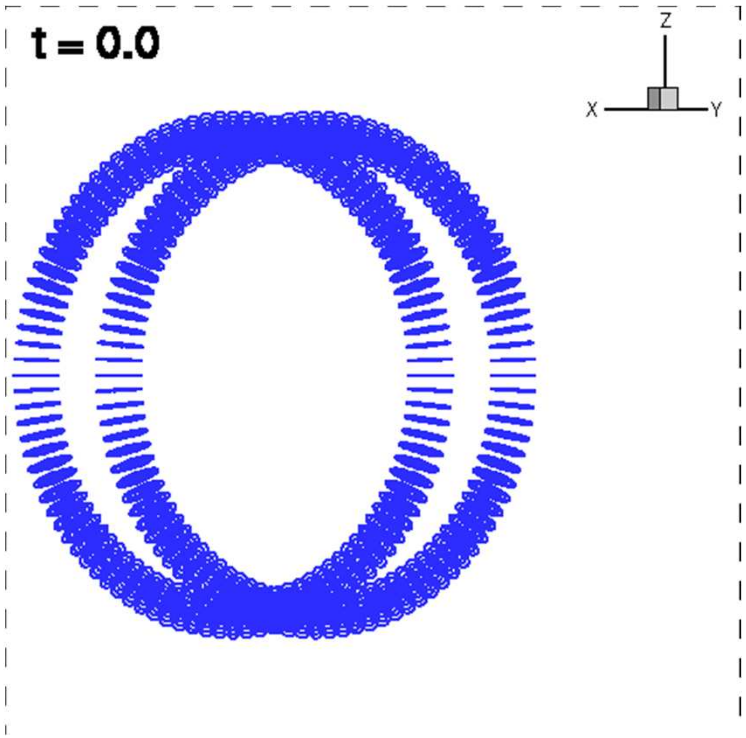
Kudela, Regucki, ICCS, 2004

Parameters:

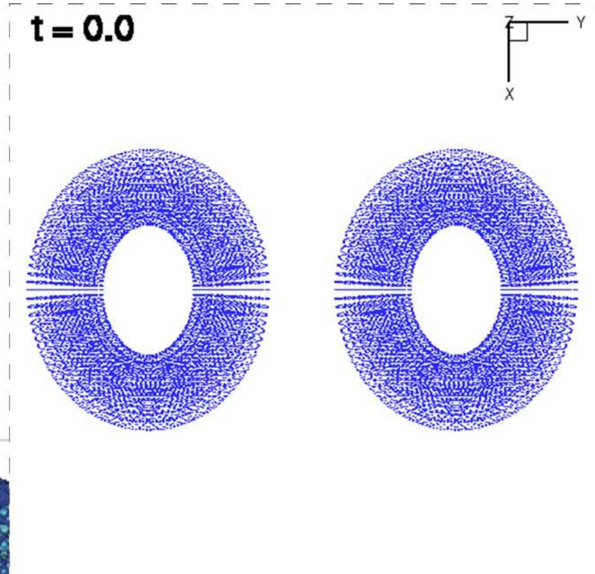
- $r_1 = 0.15$, $r_2 = 0.15$
- $R_1 = 1.5$, $R_2 = 1.5$
- $\Gamma_1 = 1.0$, $\Gamma_2 = 1.0$

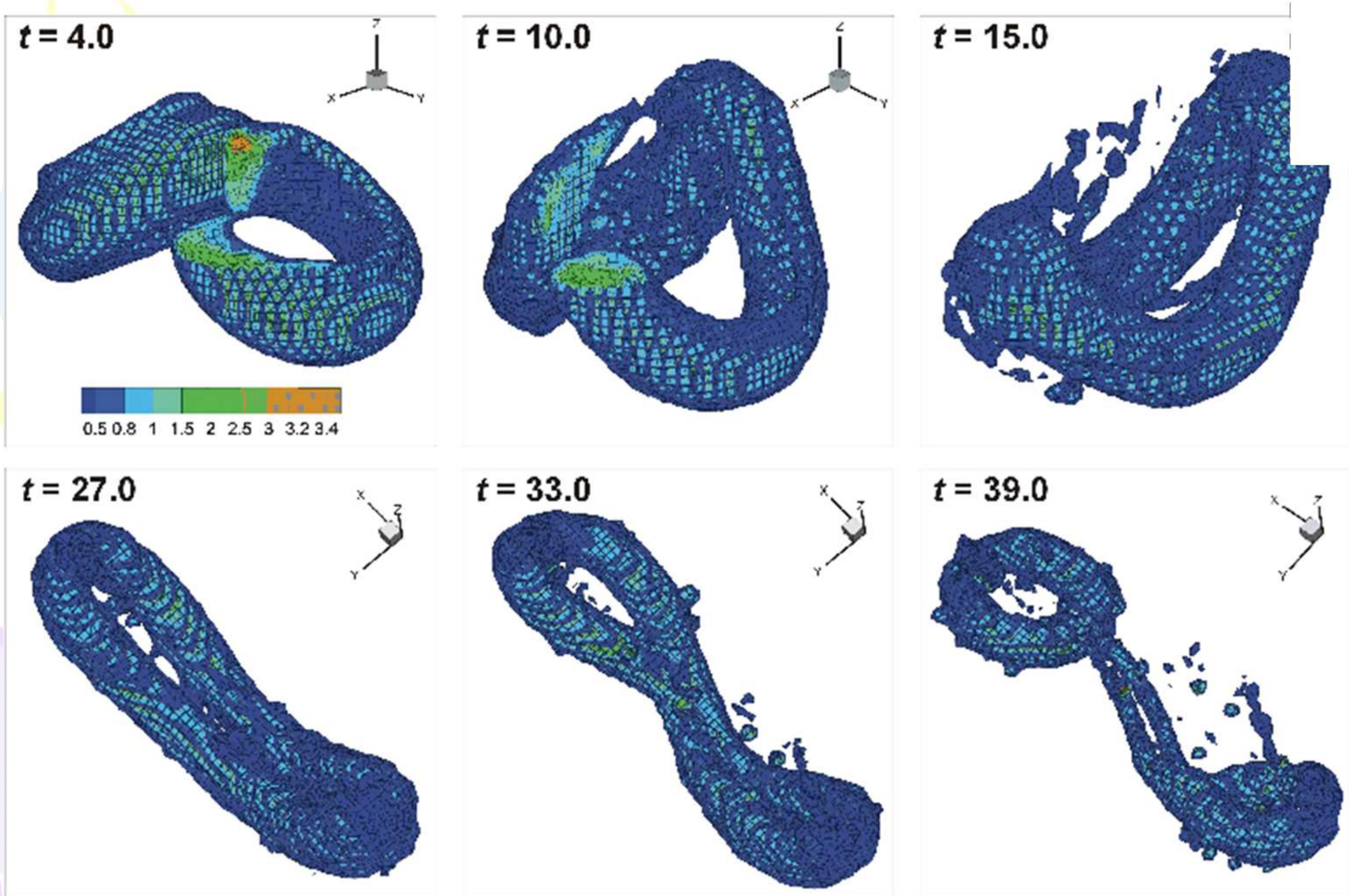
-kinetic energy: $T_0 = 5.25$ -variation~ -3%,
- helicity $H_0 = 10^{-5}$,
- divergence of A, u, ω; ~ 10^{-5},





Reconnection of vortex rings

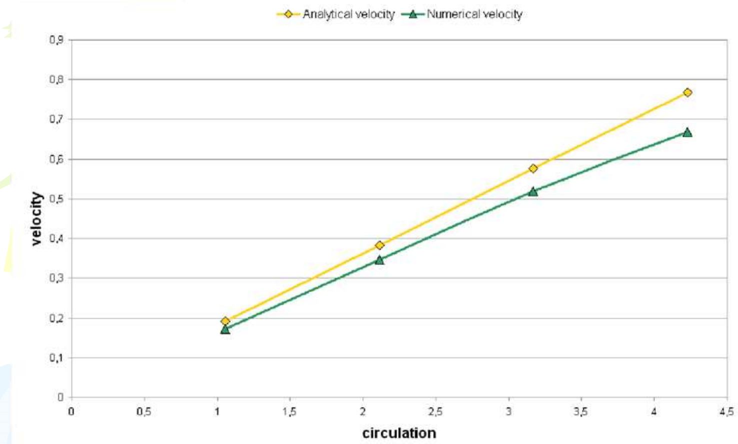




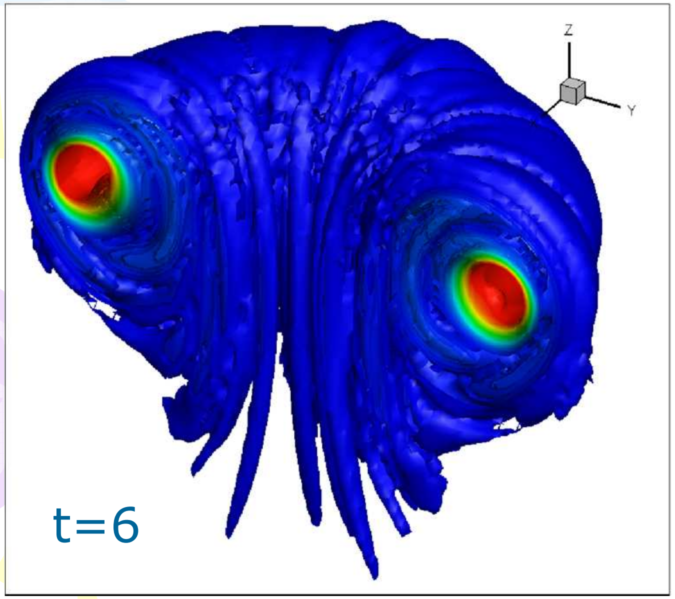
Regucki, *PhD thesis,2003*

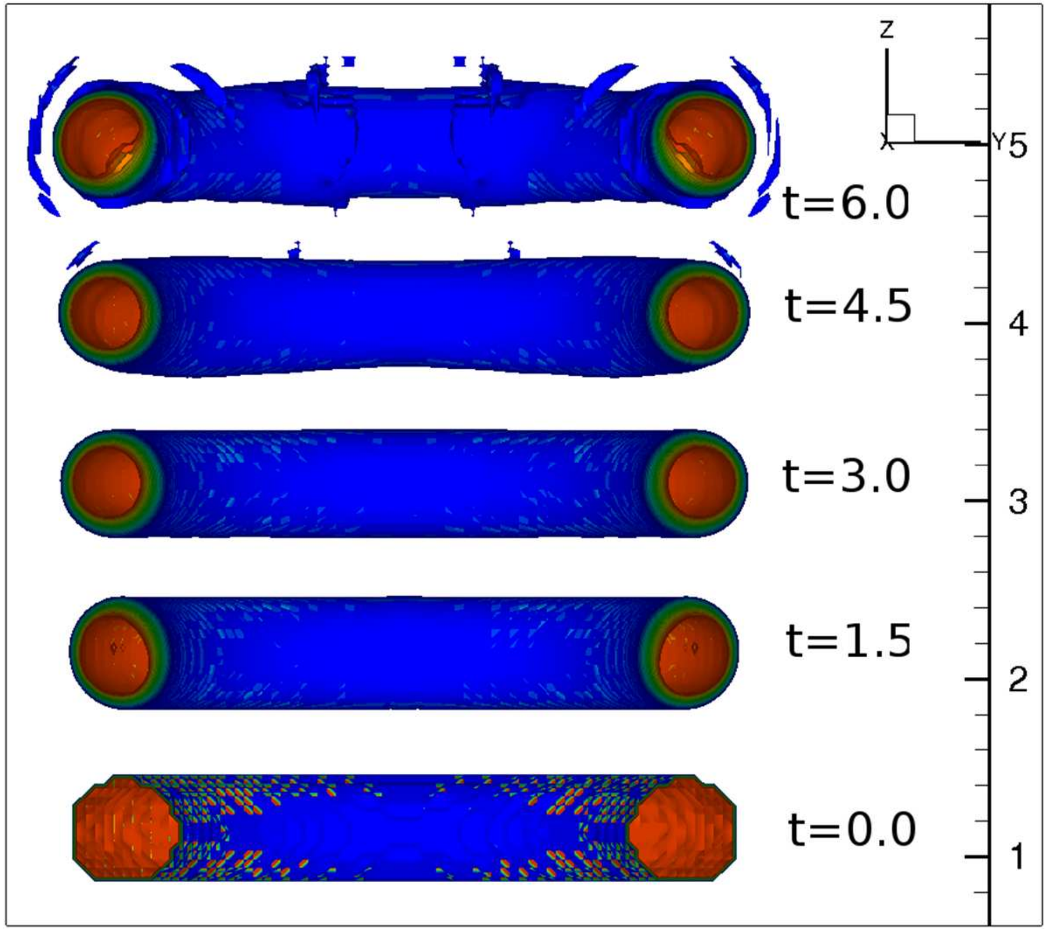
Parallel Computations – vortex ring

Andrzej Kosior,PhD. student



Analytical and numerical velocities of the vortex ring

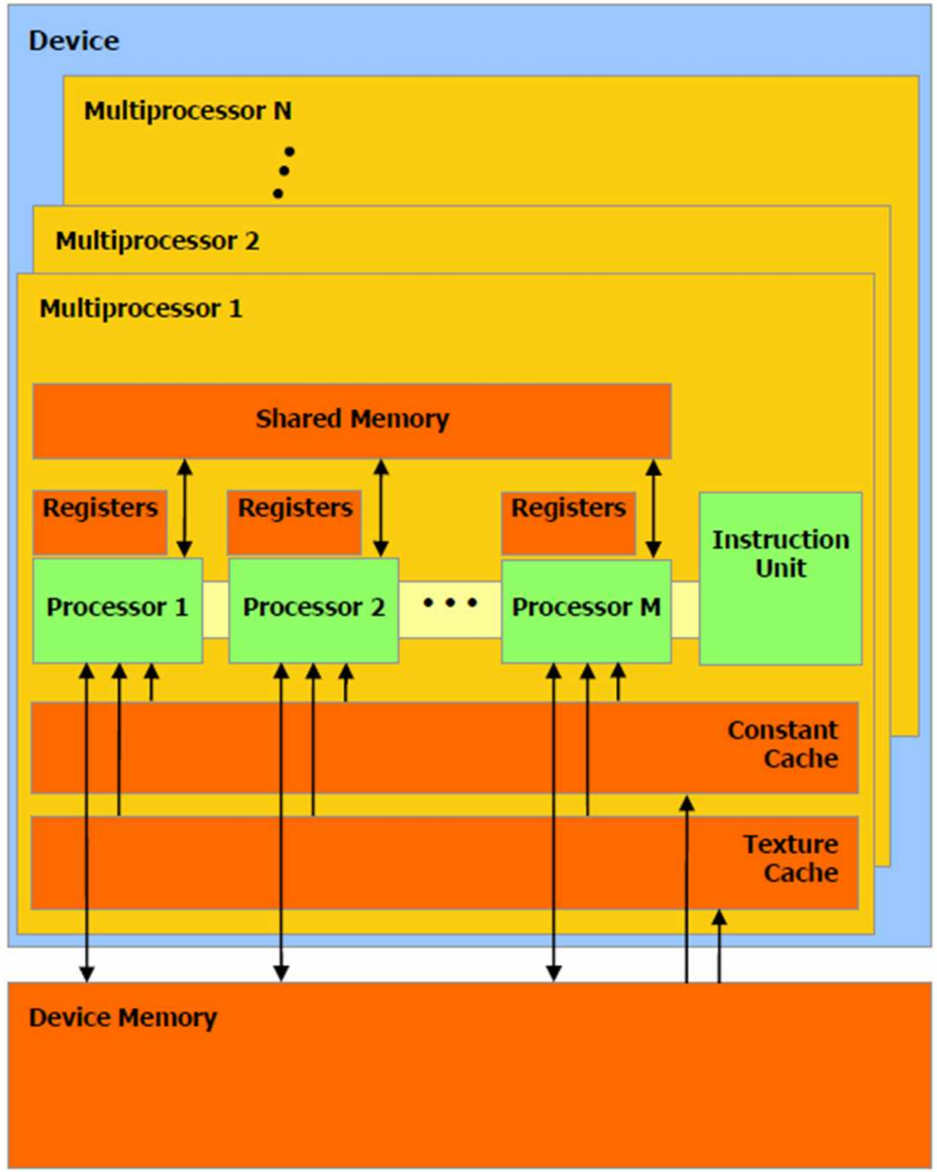




Evolution of the vortex ring

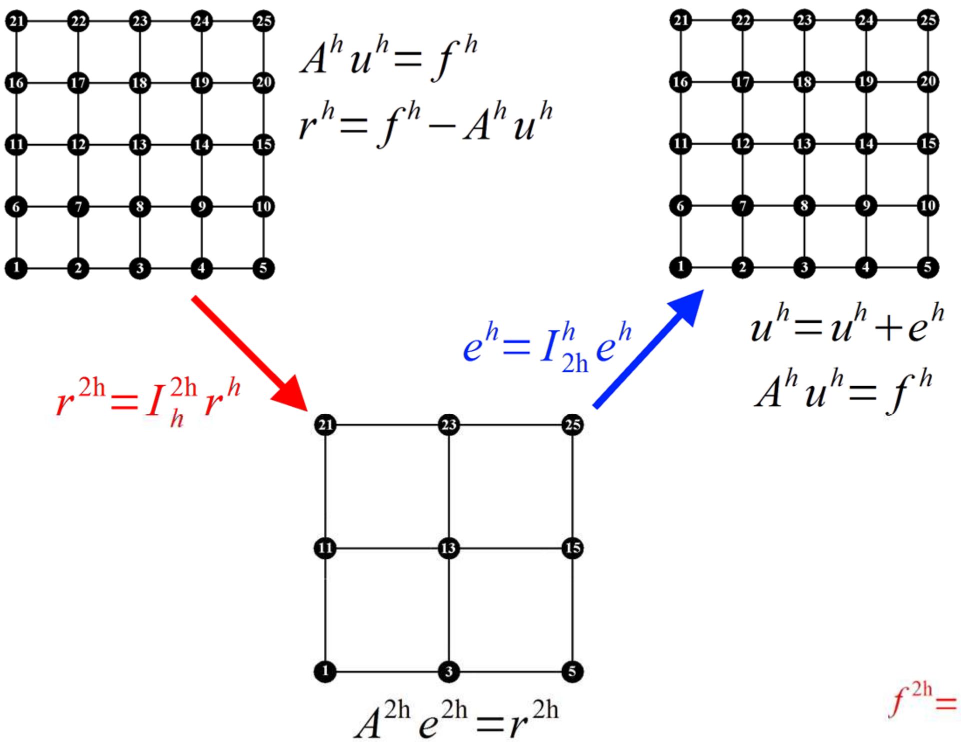
CUDA Hardware

CUDA hardware structure:
- multiprocessors,
- streaming processors,
- one instruction unit per multiprocessor,



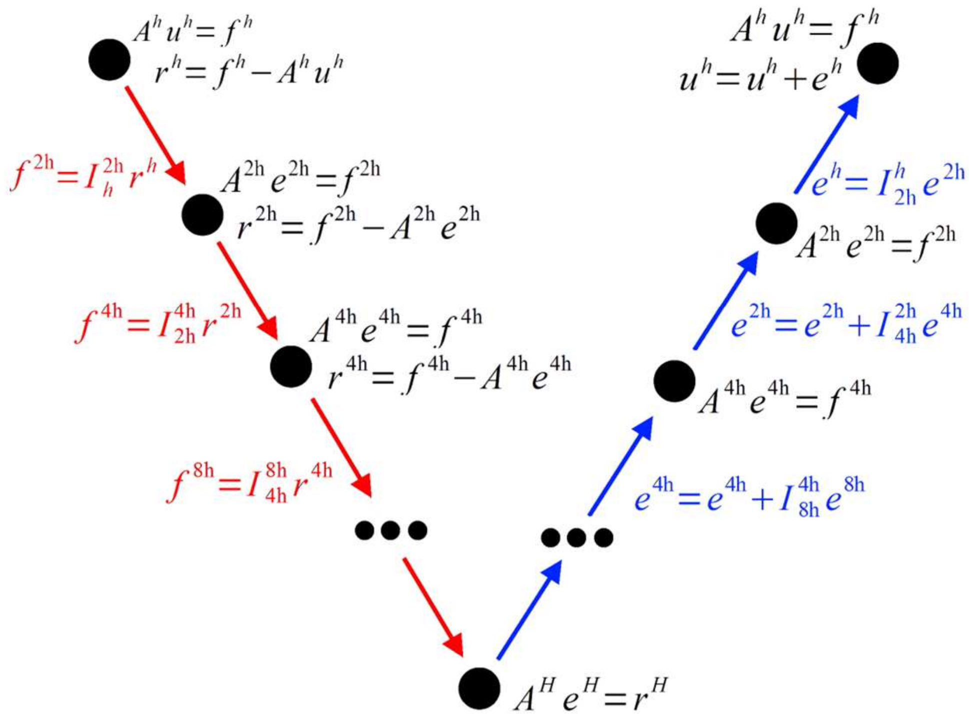
In CUDA architecture we can distinguish following memory types:
- device memory,
- texture memory,
- shared memory.

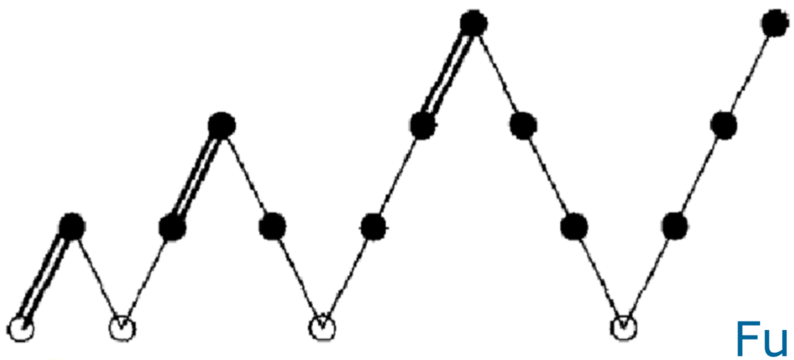
Multigrid method
For Poisson equation



Two-grid method

Multigridgrid method





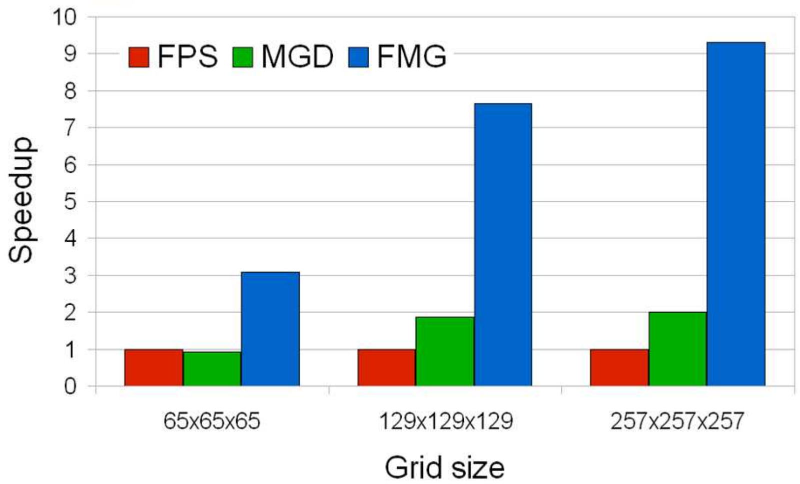
Full Mulitgrid Method

Test of Multigrid Method

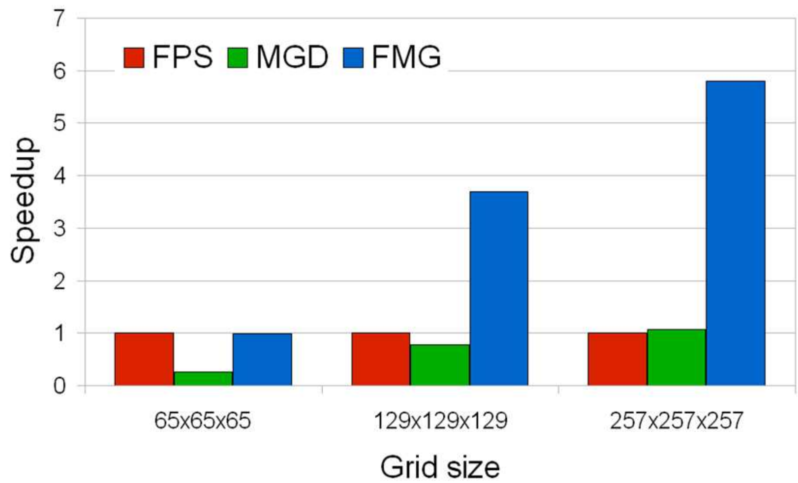
The test problem was a three-dimensional Poisson equation which solution was following function:

$$\psi(x,y,z) = sin(2\pi x)\cdot sin(2\pi y)\cdot sin(2\pi z); x,y,z \in [0,1]$$

There were two different boundary conditions tested.



Speed-up for Dirichlet boundary condition



Speed-up for periodical boundary condition

Computations were performed on:
CPU (Intel i7 960),
GPU (NVIDIA GeForce GTX 480).

Summary

- **V**ortex **M**ethods provide natural, useful tools for analyzing the flow in terms of vorticity dynamics
- **VM** are rubust and give reasonable results at a wide range of Reynolds number
- 3D **VM** are very promising for parallel computations